1 Preference and choice

1.1 Preference relations

$\succcurlyeq$ is a binary relation on the set of alternatives X. Suppose we have two alternatives

$x, y \in X$, $x \succcurlyeq y \Leftrightarrow x$ is at least as good as y.

$\succ$: the strict preference $x \succ y \Leftrightarrow x \succcurlyeq y$ but not $y \succcurlyeq x$. Read as "x is preferred to y."

$\sim$: indifference relation $x \sim y \Leftrightarrow x \succcurlyeq y$ and $y \succcurlyeq x$. Read as "x is indifferent to y."

Definition 1.B.1: The preference relation $\succ$ is rational if:
(1) Completeness: $\forall x, y \in X$, we have $x \succ y$ or $y \succ x$ (or both).
(2) Transitivity: $\forall x, y, z \in X$, if $x \succ y$ and $y \succ z$, then $x \succ z$.

Proposition 1.B.1: if $\succcurlyeq$ is rational, then:
(1) $\succ$ is irreflexive ($x \succ x$ never holds) and transitive ($x \succ y \succ z \Rightarrow x \succ z$).
(2) $\sim$ is reflexive ($x \sim x$), transitive and symmetric ($x \sim y \Rightarrow y \sim x$).
(3) $x \succ y \succcurlyeq z \Rightarrow x \succ z$.
Proof: Suppose $x \succ x$, this implies that $x \succcurlyeq x$ but not $x \succcurlyeq x$. But this is a contradiction by itself. I will leave the rest of proof to you as assignment.

1.2 Utility functions

From what you have learned before, we often describe preference relations using a utility function. A utility function $u(x)$ assigns a numerical value to each element in the choice set X.

Definition 1.B.2: A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succcurlyeq$ if, $\forall x, y \in X$,

$$x \succcurlyeq y \Leftrightarrow u(x) \geq u(y)$$

Any strictly increasing function of the original utility function $f(u(.))$ is a new representation.
Proof: Let $x, y \in X$. Since $u(.)$ represents $\succcurlyeq$, $x \succcurlyeq y$ iff $u(x) \geq u(y)$. Since $f(.)$ is strictly increasing, $u(x) \geq u(y)$ iff $f(u(x)) \geq f(u(y))$. Therefore, $x \succcurlyeq y$ iff $f(u(x)) \geq f(u(y))$. Hence, $f(u(.))$ also reprents $\succcurlyeq$

Ordinal: invariant for any strictly increasing transformation
Cardinal: not preserved for such transformation

Proposition 1.B.2: $\succcurlyeq$ can be represented by a utility function only if it is rational.
Proof: the proposition says if the preference can be represented by a utility funciton, then the preference is complete and transitive.

Completeness: since $u(.)$ is a real-valued function defined on X, it must be that $\forall x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. By the definition of utility function, this implies that either $x \succcurlyeq y$ or the other way around. Hence, $\succcurlyeq$ is complete.
Transitivity: $\forall x \succcurlyeq y \succcurlyeq z$. Since $u(.)$ represents $\succcurlyeq$, then $u(x) \geq u(y) \geq u(z)$. Therefore, $u(x) \geq u(z)$. Again, since $u(.)$ represents $\succcurlyeq$, we have $x \succcurlyeq z$. Done.

2 Consumer's choice

2.1 Commodities

Write the consumption vector

$$x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_L \end{bmatrix} \in \mathbb{R}^L,$$

2.2 The consumption set

Insert graph here.

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_l \geq 0, \forall l\}$$

2.3 Competitive budgets

Denote the price vector as

$$p = \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_L \end{bmatrix} \in \mathbb{R}^L,$$

Thus without loss of generality, we assume that $p \gg 0$**, i.e.,** $p_l > 0, \forall l$**.**

We also assume that consumers are price taker.

w: the consumers wealth level

Then, we can write down the consumer's budget constraint:

$$p \cdot x = p_1 x_1 + \cdots + p_l x_L \leq w$$

Definition: The Walrasian or competitive budget set is

$$B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$$

In most of time, the consumer will choose a point from the budget hyperplane: $\{x \in \mathbb{R}_+^L : p \cdot x = w\}$.

Insert graph here.

2.4 Demand functions and comparative statics

Walrasian (or market, or ordinary) demand correspondence $x(p, w)$ **is a set of consumption bundle the consumer may choose facing Walrasian budget set.**

If $x(p,w)$ is single-valued, it is called a demand function.

Definition 2.E.1: Homogeneous of degree zero$\Leftrightarrow x(p\alpha, w\alpha) = x(p,w), \forall p, w \ and \ \alpha > 0$**.**

Homogeneous of degree k$\Leftrightarrow x(p\alpha, w\alpha) = \alpha^k x(p,w), \forall p, w \ and \ \alpha > 0$**.**
Define $f(x,y) = 1$**, it is homogeneous of degree zero;**
$f(x,y) = x^a y^b$**, then** $f(\alpha x, \alpha y) = \alpha^a x^a \alpha^b y^b = \alpha^{a+b} f(x,y)$**. Thus, it is homogenous of degree** $a+b$**.**
what about $f(x,y) = \min x, y$**? Homogenous of degree one.**

Definition 2.E.2: $x(p,w)$ **satisfies the Warlas' law: if** $\forall p \gg 0$ **and** $w > 0$**, we have** $p \cdot x = w, \forall x \in x(p,w)$.

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ \cdot \\ \cdot \\ \cdot \\ x_L(p,w) \end{bmatrix}.$$

To sum up, the assumptions on the Walrasian demand correspondence are: homogeneous of degree zero, Warlas' law, single-valued, continuous and differentiable.

2.5 Comparative statics

2.5.1 Wealth effect

Engel function: $E_{\bar{p}} = \{x(\bar{p}, w) : w > 0\}$**.**

Insert graph here

$\frac{\partial x_l(p,w)}{\partial w}$: **wealth effect for the** l^{th} **good.**

$\frac{\partial x_l(p,w)}{\partial w} \geq 0 :$ l^{th} **good is normal at (p,w).**

$\frac{\partial x_l(p,w)}{\partial w} < 0 :$ l^{th} **good is inferior at (p,w).**

$\frac{\partial x_l(p,w)}{\partial w} \geq 0, \forall (p,w), l \Leftrightarrow$ **demand is normal.**

$$D_w x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L.$$

2.5.2 Price effects

$\frac{\partial x_l(p,w)}{\partial p_k}$: **price effect of p_k on the demand for good l.**

In matrix notation, the price effects can be written as:

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}.$$

Proposition 2.E.1: If $x(p,w)$ is homogeneous of degree zero, then

$$\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_k + \frac{\partial x_l(p,w)}{\partial w} w = 0, \forall l$$

Proof:

$$x(\alpha w, \alpha w) - x(p,w) = 0 \tag{1}$$

Differentiating both side with respect to α and evaluating $\alpha = 1$ yield the result.

We can divide both side by $x_l(p,w)$ and express the above equation as follows:

$$\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{x_l(p,w)} + \frac{\partial x_l(p,w)}{\partial w} \frac{w}{x_l(p,w)} = 0, \forall l$$

This part is the elasticities of demand with respect to prices and wealth, denoted by $\epsilon_{lk}(p,w)$ and $\epsilon_{lw}(p,w)$. Thus,

$$\sum_{k=1}^{L} \epsilon_{lk}(p,w) + \epsilon_{lw}(p,w) = 0, \forall l$$

An equal percentage change in all prices and wealth leads to no change in demand.

Proposition 2.E.2: If $x(p, w)$ satisfies Walras' law, then

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial p_k} + x_k(p, w) = 0, \forall k, p, w$$

Proposition 2.E.4: If $x(p, w)$ satisfies Walras' law, then,

$$\sum_{l=1}^{L} p_l \frac{x_l(p, w)}{\partial w} = 1, \forall p, w$$

3 Classical demand theory

3.1 Preference relations: Basic properties

Definition 3.B.1: $\succsim$ is rational if

(i) Completeness. $\forall x, y \in X$**,** $x \succsim y$ **or** $y \succsim x$ **or both.**

(ii) Transitivity. $\forall x, y, z \in X$**,** $x \succsim y$ **and** $y \succsim z \Rightarrow x \succsim z$**.**

Definition 3.B.2: $\succsim$ **is monotone if** $x \in X$ **and** $y \gg x \Rightarrow y \succ x$**.**

$\succsim$ **is strongly monotone if** $y \geq x$ **and** $y \neq x \Rightarrow y \succ x$**.**

Definition 3.B.3 $\succsim$ **is locally non-satiated if** $\forall x \in X$ **and** $\epsilon > 0$**,** $\exists\ y \in X$**, such that** $||y - x|| \leq \epsilon$ **and** $y \succ x$. $||x - y|| = [\sum_{l=1}^{L}(x_l - y_l)]^{0.5}$ is Euclidean distance.

Insert graph here.

Indifference set of x**:** $\{y \in X : y \sim x\}$

Upper contour set of x**:** $\{y \in X : y \succsim x\}$

Lower contour set of x**:** $\{y \in X : x \succsim y\}$

Insert graph here.

If $\succcurlyeq$ is strongly monotone. Pick up $y \gg x$. Then $y \geq x$ and $y \neq x$, strongly monotonicity implies that $y \succ x$. Therefore, $\succcurlyeq$ is monotone.
Assume that $\succcurlyeq$ is monotone, $x \in X$ and $\epsilon > 0$. Let $e = (1, \cdots, 1) \in \mathbb{R}_+^{\mathbb{L}}$ and $y = x + (\epsilon/\sqrt{L})e$. Then $||y - x|| \leq \epsilon$ and $y \succ x$. Thus $\succcurlyeq$ is locally nonsatiated.

Definition 3.B.4: $\succcurlyeq$ is convex if $\forall x \in X$, its upper contour is convex; i.e., if $y \succcurlyeq x$ and $z \succcurlyeq x$, then $\alpha y + (1-\alpha)z \succcurlyeq x, \forall \alpha \in [0, 1]$.

Insert graph here.

Definition 3.B.5: $\succcurlyeq$ is strictly convex if $y \succcurlyeq x$, $z \succcurlyeq x$, $y \neq z \Rightarrow$ $\alpha y + (1-\alpha)z \succ x, \forall \alpha \in (0, 1)$.

Insert graph here.

Definition 3.B.6: $\succsim$ **is homothetic if** $x \sim y$**, then** $\alpha x \sim \alpha y, \forall \alpha \geq 0.$

Insert graph here.

Definition 3.B.5: $\succsim$ **on** $X = (-\infty, +\infty) \times \mathbb{R}_+^{L-1}$ **is quasilinear with respect to commodity 1 (numeraire commodity) if (i)** $x \sim y \Rightarrow x + \alpha e_1 \sim y + \alpha e_1$**, where** $e_1 = (1, 0, \cdots, 0)$**, (ii) commodity 1 is desirable:** $x + \alpha e_1 \succ x, \forall x, \forall \alpha > 0$**.**

Insert graph here

3.2 Preference and Utility

Example: Consider the Lexicographic preference relation. $X = \mathbb{R}_+^2$**. Consider two alternatives:** $x = (x_1, x_2)$ **and** $y = (y_1, y_2)$**. Define the preference** $x \succsim y \Leftrightarrow$ **"**$x_1 \succ x_1 > y_1$**" or "**$x_1 = y_1$ **and** $x_2 \geq y_2$**". This preference implies that commodity has the highest priority in determining the preference ordering.**

Insert the graph here.

Definition 3.C.1: $\succsim$ on X is continuous if it preserves under limit. For all sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n, \forall n$, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.

Example: Consider the sequence of bundles $x^n = (1/n, 0)$ and $y^n = (0, 1)$. $\forall n$, we have $x^n \succ y^n$. We know $\lim_{n\to\infty} x^n = (0, 0)$ and $\lim_{n\to\infty} y^n = (0, 1)$. Thus, $y \succ x$. But this contradicts the continuity.

Proposition ((Jehle and Reny Theorem 1.1): If $\succsim$ on $X = R^L_+$ is rational, strictly monotone and continuous, then $\exists$ a continuous utility function $u(x)$ that represents $\succsim$.

Proof: Step 1: Let $e \equiv (1, \cdots, 1) \in \mathbb{R}^L_+$ denote a unit consumption bundle (which contains one unit of each good). For any consumption bundle x, we want to find $u(x)$ such that

$$u(x)e \sim x$$

$u(x)e$ is the consumption bundle that consists of $u(x)$ units of each good. Therefore, we need the existence of such $u(x)$

Step 2: By continuity of preferences, the sets

$$A^+ \equiv \{t \in \mathbb{R}_+ : te \succsim x\}$$

and

$$A^- \equiv \{t \in \mathbb{R}_+ : x \succsim te\}$$

are both nonempty and closed. Why there are both nonempty? We know at least $0 \cdot e \in A^-$ and for $u(x)$ large enough, it belongs to A^+. By completeness of $\succsim$, we know $\mathbb{R}_+ \subset (A^+ \cup A^-)$. By non-emptiness and closedness of A^+ and A^- and $\mathbb{R}_+$ is connected, we know $A^+ \cap A^- \neq \emptyset$. What is $A^+ \cap A^-$? It is the set $A^+ \cap A^- \neq \{t \in \mathbb{R}_+ : te \sim x\}$. Thus, there exists a scalar such that $u(x) \sim x$. Furthermore, the monotonicity implies that $\alpha_1 e \succ \alpha_2 e \Leftrightarrow \alpha_1 > \alpha_2$. Thus the scalar is unique.

Step 3. For any two consumption bundles x and x',

$$x \succsim x' \Leftrightarrow u(x)e \succsim u(x')e \Leftrightarrow u(x) \geq u(x').$$

The first step follows the definition of $u(.)$ and the second step follows the monotonicity assumption. Hence $u(\cdot)$ represents the preference.

Step 4. Continuity. Skipped. If you are interested in this, you can read the book. And if you have any question, you can came to me.

Proposition 3.C.1: If $\succsim$ is rational and continuous, then $\exists$ a continuous utility function $u(x)$ that represents $\succsim$.

Remark 1. Any strictly increasing transformation of $u(\cdot)$ also represents the pref-

erence.

Remark 2. Some discontinuous function can also represent the preference.

Monotonicity of preference $\Rightarrow$ the utility function is increasing: $u(x) > u(y)$ if $x \gg y$.

(Strictly)Convexity of preference $\Rightarrow$ $u(.)$ is (strictly) quasiconcave.

Definition: The utility function $u(.)$ is quasiconcave if $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex $\forall x$ or equivalently, if $u(\alpha x + (1-\alpha)y) \geq Min\{u(x), u(y)\}, \forall x, y \in X, \alpha \in [0,1]$.

Remark: Convexity of preference does not imply $u(.)$ is concave.

Homothetic of preference $\Rightarrow$ the utility function is homogeneous of degree one, i.e., $u(\alpha x) = \alpha u(x), \forall \alpha > 0$.

Quasilinearity of preference $\Rightarrow$ the utility function takes the form of $u(\alpha x) = x_1 + \phi(x_2, \cdots, x_l)$.

3.3 The utility maximization problem (UMP)

Here is consumer's problem

$$\max_{x \geq 0} \; u(x)$$

s.t. $p \cdot x \leq w$

Proposition 3.D.1: If $p \gg 0$ and $u(\cdot)$ is continuous, then the UMP has a solution.

Insert the graph.

Proposition 3.D.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succcurlyeq$ on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p,w)$ possesses the following properties:
(i) Homogeneity of degree zero in (p,w): $x(\alpha p, \alpha w) = x(p,w), \forall p, w, \alpha > 0.$
(ii)Walras' law: $p \cdot x = w, \forall x \in x(p,w)$.
(iii)If $\succcurlyeq$ is convex, so that $U(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set. If $\succcurlyeq$ is convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ is singleton.

Proof: (i) If we look at the UMP. For any $\alpha > 0$, the objective function is $U(x)$; the

feasible set is $\alpha p \cdot x \leq \alpha w \Leftrightarrow p \cdot x \leq w$. Therefore, the solution for the UMP does not depend on α, i.e., $x(p,w) = x(\alpha p, \alpha w)$.

(ii)**Recall the properties of locally-nonsatiated. It means that for any alternative x, there exists another alternative y, which is close to x and preferred to x. Suppose not and $\exists x \in x(p,w)$ such that $p \cdot x < w$. This implies that $\exists$ another consumption bundle y which is sufficiently close to x such that $p.y < w$ and $y \succ x$. This means y is feasible but is better that x. This contradicts x being optimal.**

(iii) We need to show that $\forall, x, x' \in x(p,w)$ with $x \neq x'$, $x'' = \alpha x + (1-\alpha)x' \in x(p,w), \forall \alpha \in [0,1]$. We know that $u(x) = u(x')$ due the fact that x, x' both solves the UMP. Denote the utility level by u^*. By quasiconvavity of $u(x)$, we know $u(x'') \geq \alpha u(x) + (1-\alpha)u(x') = u^*$. In addition, since $p.x \leq w$ and $p.x' \leq w$, we have $p.x'' = p[\alpha x + (1-\alpha)x'] \leq w$. Therefore, x'' is a feasible choice in the UMP. In sum, $u(x'') \geq u^*$ and x'' is feasible. Therefore, $x'' \in x(p,w)$.

Suppose now that $u(\cdot)$ is strictly quasiconcave. With the same argument, we can conclude that $u(x'') > u^*, \forall \alpha \in (0,1)$ and x'' is feasible. But this contradicts the assumption that x and x' solves the UMP. As a result, there can only be one element in $x(p,w)$.

$$\max_x U(x);$$

s.t.

$$p.x = w,$$

$$x \geq 0$$

The Lagrangeian function is

$$\mathcal{L} = u(x) + \lambda(w - p_1 x_1 - \cdots - p_L x_L) + \sum_{l=1}^{L} \mu_l x_l$$

The Kuhn-Tuck implies that the necessary condition for the optimality is

$$u_l(x) - \lambda p_l + \mu_l = 0, \forall l$$

$$w - p_1 x_1 - \cdots - p_L x_L = 0$$

$$\mu_l \geq 0, \forall l$$

$$x_l \geq 0, \forall l$$

$$\mu_l x_l = 0$$

If we have an interior solution x^*, then $\mu_l = 0, \forall l$. Thus $u_l(x^*) = \lambda p_l, \forall l$. From here, we

can get $\underbrace{\frac{\mu_l(x^*)}{\mu_k(x^*)}}_{marginal\ rate\ of\ substitution\ of\ good\ l\ for\ good\ k, MRS_{lk}} = \frac{p_l}{p_k}$**.** Insert graph here

Example: Consider the Cobb-Douglas Utility function with two commodities: $u(x_1, x_2) = kx_1^{\alpha}x_2^{1-\alpha}$ **with** $\alpha \in (0,1), k > 0$**. Since transfer the utility function with a strictly increasing function will not change the optimal solution. We can transform it to** $\alpha ln x_1 + (1-\alpha) ln x_2$**. Thus the UMP is**

$$\max_{x_1, x_2} \alpha ln x_1 + (1-\alpha) ln x_2$$

s.t.

$$p_1x_1 + p_2x_2 = w$$

$$x_1 \geq 0, x_2 \geq 0$$

Thus, the Lagrangian function is:

$$\mathcal{L} = \alpha ln x_1 + (1-\alpha) ln x_2 + \lambda[w - p_1x_1 - p_2x_2] + \mu_1 x_1 + \mu_2 x_2$$

The Kuhn-Tuck condition is:

$$\alpha/x_1 - \lambda p_1 + \mu_1 = 0$$

$$(1-\alpha)/x_2 - \lambda p_2 + \mu_2 = 0$$

$$p_1x_1 + p_2x_2 = w$$

$$x_1 \geq 0, x_2 \geq 0$$

$$\mu_1 \geq 0, \mu_2 \geq 0$$

$$\mu_1 x_1 = 0$$

$$\mu_2 x_2 = 0$$

First what is the interior solution. That means $\mu_1 = \mu_2 = 0$**. Therefore, we end up with**

$$\alpha/x_1 - \lambda p_1 = 0$$

$$(1-\alpha)/x_2 - \lambda p_2 = 0$$

$$p_1 x_1 + p_2 x_2 = w$$

We have three unknown variables x_1, x_2, λ and three unknowns. We can solve it as $x_1 = \frac{\alpha w}{p_1}$ and $x_2 = \frac{(1-\alpha)w}{p_2}$
What about the corner solution. Suppose $x_1 = 0$, then x_2 is finite, then $\mu_2 = 0$, then λ is finite. But then the first equation cannot hold.

$v(p,w) = u(x(p,w))$ **is called the indirect utility function.**

Proposition: Suppose $u(\cdot)$ is continuous and represents a locally non-satiated preference $\succcurlyeq$ on $X = \mathbb{R}^L_+$, then the indirect utility function $v(p,w)$ is
(i) Homogeneous of degree zero in (p,w), i.e., $v(p,w) = v(\alpha p, \alpha w), \forall \alpha > 0$,
(ii)Strictly increasing in w and nonincreasing in $p_l, \forall l$,
(iii)Quasiconvex, i.e., $\{(p,w) : v(p,w) \leq \overline{v}\}$ is convex $\forall \overline{v}$,
(iiii) Continuous in p and w.
Proof: The first one is obvious since the feasible set are the same for any α. The second one is also obvious since $p.x \leq w$, if w is higher or any price is lower, the feasible set is larger and the consumer can achieve higher utility. For (iii), suppose $v(p,w) \leq \overline{v}$ and $v(p',w') \leq \overline{v}$, we need to prove that $v(p'',w'') \leq \overline{v}$, where $(p'',w'') = (\alpha p + (1-\alpha)p'), (\alpha w + (1-\alpha)w')$. It is sufficient to prove that, given the price and wealth (p'',w''), any feasible bundle x: $\alpha p.x + (1-\alpha)p'.x \leq \alpha w + (1-\alpha)w'$, cannot generate more revenue than $\overline{v}$. The budget constraint implies that either $p.x \leq w$ or $p'.x \leq w'$ (or both), i.e., any feasible bundle under (p'',w'') can be either feasible under (p,w) or (p',w'). But any feasible bundle under $p.x \leq w$ or $p'.x \leq w'$ cannot generate more revenue than $\overline{v}$. Therefore, any feasible set under (p'',w'') cannot generate more than $\overline{v}$.

Insert the graph here.

3.4 The Expenditure Minimization Problem (EMP)

Now look at the EMP

$$\min_{x \geq 0} p.x$$

s.t. $u(x) \geq u$

Proposition: Suppose that $u(.)$ is continuous and represents a locally nonsatiated preference on $X = \mathbb{R}^L_+$. The expenditure function $e(p, u)$ is
(i) Homogeneous of degree on in p.
(ii) Strictly increasing in u and nondecreasing in $p_l, \forall l$.
(iii) Concave in p.
(iv) Continuous in p and u.
Proof: (i)

$$e(\alpha p, u) = \{min_{x \geq 0} \alpha p.x, s.t.u(x) \geq u\} \tag{2}$$

$$= \alpha\{min_{x \geq 0} p.x, s.t.u(x) \geq u\} \tag{3}$$

$$= \alpha e(p, u) \tag{4}$$

(ii) Suppose that $e(p, u)$ were not strictly increasing in u, then there exists $u' < u''$ such that $e(p, u') \geq e(p, u'')$. Denote the optimal bundles for u' and u'' as x' and x''. Therefore, $p.x' \geq p.x'' > 0$. Now consider a bundle $\tilde{x} = \alpha x''$ where $\alpha \in (0, 1)$. By continuity of $u(.)$, there exists an α close enough to 1 such that $u(\tilde{x}) > u'$ and $p.x' > p.\tilde{x}$. But this contradicts x' being optimal in EMP with u' as $\tilde{x}$ is better.
To show $e(p, u)$ is nondecreasing in p_l. Consider the price vector p'' and p' with $p''_l \geq p'_l$ and $p''_k = p'_k, \forall k \neq l$. Let x'' solves EMP at p''. Then, $e(p'', u) = p''.x'' \geq p'.x'' \geq e(p', u)$. The last inequality follows the definition of $e(p', u)$ since x' is optimal under (p', u).
(iii) For concavity, let $p' = \alpha p + (1 - \alpha)p', \alpha \in [0, 1]$.

$$e(p'', u) = p''.x'' = \alpha p.x'' + (1 - \alpha)p'.x'' \geq \alpha e(p, u) + (1 - \alpha e(p', u))$$

. The inequality follows the definition of the expenditure function.

Proposition: Suppose that $u(.)$ is continuous and represents a locally nonsatiated preference on $X = \mathbb{R}^L_+$. Hicksian demand $h(p, u)$ is
(i) Homogeneous of degree zero in p, i.e., $h(\alpha p, u) = h(p, u)$.
(ii) No excess utility: $u(h(p, u)) = u$
(iii) Convex/uniqueness: If $\succcurlyeq$ is convex, then $h(p, u)$ is a convex set; and if $\succcurlyeq$ is strictly convex, then there is a unique element in $h(p, u)$
(i)Obvious, since the optimal vector when minimizing $p.x$ subject to $u(x) > u$ is the same as that for minimizing $\alpha p.x$ subject to the same constraint.
(ii)Suppose there exists $x \in h(p, u)$ such that $u(x) > u$. Consider a bundle $x' = \alpha x$, where $\alpha \in (0, 1)$. By continuity, for α close enough to 1, $u(x') \geq u$ and $p.x' < p.x$. But this contradicts x being optimal in the EMP, since x' is better.
(iii) I will the proof as an assignment as it is similar to the proof for the Warlasian Demand function.

Proposition 3.E.4: Suppose that $u(.)$ is continuous and represents a strictly convex locally nonsatiated preference on $X = \mathbb{R}^L_+$. $p \gg o$. Then $h(p,u)$ satisfies:

$$(p'' - p').[h(p'', u) - h(p', u)] \leq o$$

Proof: $\forall p \gg 0$, **$h(p,u)$ is optimal in EMP:**

$$p''(p'', u) \leq p''h(p', u)$$

$$p'.h(p'', u) \geq p'.h(p', u)$$

(1)-(2) yields the result.

$$\min_{x \geq 0} p.x$$

s.t. $u(x) = u$

$$\mathcal{L} = p.x + \lambda(u(x) - u) + \sum_{l=1}^{L} \mu_l x_l$$

The Kuhn-Tuck condition is

$$p_l + \lambda \frac{\partial x u(x)}{\partial x_l} + \mu_l = 0$$

$$u(x) - u = 0$$

$$x_l \geq 0, \mu_l \geq 0$$

$$\mu_l x_l = 0$$

If the solution is interior, $u_l(x^*) = \lambda p_l, \forall l$. From here, we can get

$$\underbrace{\frac{\mu_l(x^*)}{\mu_k(x^*)}}_{marginal\ rate\ of\ substitution\ of\ good\ l\ for\ good\ k, MRS_{lk}} = \frac{p_l}{p_k}.$$

3.5 Relationships between demand, indirect utility and expenditure functions

Insert graph here.

Proposition 3.E.1: Suppose that $u(.)$ is continuous and represents a locally nonsatiated preference on $X = \mathbb{R}^L_+$. We have
(i) if x^* is optimal in the UMP when the wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure in this EMP is exactly w.
(ii) If x^* is optimal in the EMP when the required utility is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p.x^*$. Moreover, the maximized utility level in this UMP is exactly u.
Proof: (i) Suppose x^* is not optimal in the EMP with required utility $u(x^*)$. Then there exists an x' such that $u(x') \geq u(x^*)$ and $p.x' < p.x^* \leq w$. By local nonsatiation, we can find an x'' very close to x' such that $u(x'') > u(x') \geq u(x^*)$ and $p.x'' < w$. But this implies that x'' better that x'' is better in UMP. Contradiction. Therefore, x^* must be optimal in the EMP with $u(x^*)$, and the minimized expenditure level is $p.x^*$. Finally, since x^* solves the UMP with w, by Warlas' law we have $p.x^* = w$.
(ii) Since $u > u(0)$, we must have $x^* \neq 0$. Hence, $p.x^* > 0$. Suppose that x^* is not optimal in the UMP when wealth is $p.x^*$. Then there exists an x' such that $u(x') > u(x^*)$ and $p.x' \leq p.x^*$. Consider a bundle $x'' = \alpha x', \alpha \in (0, 1)$. By continuity of $u(.)$, if α is close enough to 1, then we will have $u(x'') > u(x^*)$ and $p.x'' < p.x^*$. This means that x'' is better than x^* in EMP. Contradiction. Thus, x^* must be optimal in UMP when wealth is $p.x^*$, and the maximized utility level is therefore $u(x^*)$. By no excess utility property, $u(x^*) = u$.

$$e(p, v(p, w)) = w, v(p, e(p, u)) = u$$

$$h(p, u) = x(p, e(p, u)), x(p, w) = h(p, v(p, w))$$

3.5.1 Hicksian Demand and the expenditure function

Proposition: $u(.)$ is continuous and represents a locally nonsatiated and strictly convex preference $\succcurlyeq$ on $\mathbb{R}^L_+$, then

$$h_l(p, u) = \frac{\partial e(p, u)}{\partial p_l}, \forall l$$

or

$$h(p, u) = \nabla_p e(p, u)$$

Proof: $e(p,u) = \min p.x$**, s.t.** $u(x) \geq u = \min p.x$**, s.t.** $u(x) = u$**.**

$$\mathcal{L} = p.x + \lambda(u(x) - u)$$

From the Envelope theorem: $\frac{\partial e(p,u)}{\partial p_l} = \frac{\partial \mathcal{L}}{\partial p_l}\Big|_{x=h(p,u)} = h(p,u).$

3.5.2 The Hicksian and Marshallian demand function

In reality, Marshallian demand function is usually observable, while Hicksian demand function is not since u is not. However, we can compute Hicksian demand from Marshallian demand, this is the well-know Slutsky equation.

Proposition: $u(.)$ **is continuous and represents a locally nonsatiated and strictly convex preference** $\succcurlyeq$ **on** $\mathbb{R}^L_+$**. Then For all** (p,w) **and** $u = v(p,w)$**, we have**

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w), \forall l,k.$$

Proof: We know $h_l(p,u) = x_l(p,e(p,u))$**. Differentiating both side with respect to** p_k **yields**

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} \underbrace{\frac{\partial e(p,u)}{\partial p_k}}_{=h_k(p,u)=x_k(p,w)}, \forall l,k.$$

3.5.3 Marshallian demand and the indirect utility function

Can we obtain similar relationship for Marshallian demand and the indirect utility function? This relationship is called the Roy's identity.

Proposition: $u(.)$ **is continuous and represents a locally nonsatiated and strictly convex preference** $\succcurlyeq$ **on** $\mathbb{R}^L_+$**. Suppose that the indirect utility function is differentiable at** $(\bar{p},\bar{w}) \gg 0$**. Then**

$$x_l(\bar{p},\bar{w}) = -\frac{\partial v(\bar{p},\bar{w})/\partial p_l}{\partial v(\bar{p},\bar{w})/\partial w}$$

Proof: here we look at the envelope theorem. $v(p,w) = \max\ u(x), s.t.\ p.x = w$**, thus,**

$$\mathcal{L} = u(x) + \lambda(w - p.x)$$

.

$$\frac{\partial v(p,w)}{\partial p_l} = \frac{\partial \mathcal{L}}{\partial p_l} = -\lambda x_l$$

$$\frac{\partial v(p,w)}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} = w$$

Therefore, we have the result.

4 Production Theory

4.1 Production set

For example: $y = (-1, -5, -10, 6, 1)$ it means in the production, we input 1 unit of commodity one, 5 units of commodity two and then units of commodity three to produce six units of commodity four and one unit of commodity five.

Production set $Y \subset \mathbb{R}^{\mathbb{L}}$: the set of all production vectors that constitute feasible plans for the firm
It is sometimes convenient to describe the production set using a function
$F(\cdot)$ **called the transformation function:** $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$ **and** $F(y) = 0$ **iff** y **is an element of the boundary of Y.**
Transformation frontier: the set of boundary points of Y**:** $\{y \in \mathbb{R}^L : F(y) = 0\}$
Here is an example for the two dimentional case:

For exampe: If $F(y_1, y_2) = y_1 + 2y_2$,

For the two dimentional case, from the frontier $F(y_1, y_2) = 0$, we can take the total differentiation:

$$F_1(y_1, y_2)dy_1 + F_2(y_1, y_2)dy_2 = 0 \Leftrightarrow \frac{dy_2}{dy_1} = -\frac{F_1}{F_2} = -MRT_{12}$$

MRT_{12}:

4.2 Properties of Production sets

(i) Y is nonempty I believe this is reasonable, otherwise, the firm has nothing to do and it is of no interest.

(ii)Y is closed: $y^n \in Y \Rightarrow \lim_{n \to \infty} \in Y$. That means the production set includes its boundary.

(iii)No free lunch: $y \geq 0, y \in Y \Rightarrow Y = 0$

(iv)Possibility of inaction: $y = 0 \in Y$ Complete shutdown is possible.

(v) Free disposal: $y \in Y$ **and** $y' \leq y \Rightarrow y' \in Y$ Extra amount of inputs or outputs can be eliminated at no cost.

(vi)Irreversibility: $y \in Y$ **and** $y \neq 0 \Rightarrow -y \ not \ \in Y$

(vii)Nonincreasing returns to scale: $y \in Y \Rightarrow \alpha y \in Y, \forall \alpha \in [0, 1]$

(viii)Nondecreasing returns to scale: $y \in Y \Rightarrow \alpha y \in Y, \forall \alpha \geq 1$ Any feasible input-output vector can be scaled up.

(ix)Constant returns to scale: $y \in Y \Rightarrow \alpha y \in Y, \forall \alpha \geq 0$

(x) Additivity (free entry): $y \in Y, y' \in y \Rightarrow y + y' \in Y$

(xi) Convexity: Y **is convex, i.e.,** $\alpha y + (1-\alpha)y' \in Y, \forall y, y' \in Y, \alpha \in [0,1]$**.**

(xii) Y is a convex cone: convexity + constant return to scale. Formally, Y is a convex cone iff $\alpha y + \beta y' \in Y, \forall \alpha, \beta \geq 0, y, y' \in Y$

4.3 Profit maximization and cost minimization

price the firm faces be $p = (p_1, \cdots, p_L) \gg 0$. That means there are in total L inputs and outputs. We have discussed many properties about the production set above, we always **assume that** Y **has the properties of nonemptiness, closedness, and free disposal.**

4.3.1 The profit maximization problem (PMP)

$$max_y p.y$$

s.t. $y \in Y$

$$max_y \ p.ys.t.F(y) \leq 0$$

$$\max_z \ py - w.zs.t.y = f(z)$$

Insert graph here.

Insert graph here

Proposition 5.C.1: Assume Y is closed and satisfies the free disposal property, then
(i) $\Pi(p)$ is homogeneous of degree one in p.
(ii) $\Pi(p)$ is convex in p.
(iii) $y(p)$ is homogeneous of degree zero in p.
(iv) If Y is convex, then $y(p)$ is a convex set. Moreover, if Y is strictly convex, then $y(p)$ is signle-valued (if nonempty).
(v) (Hotelling's lemma) If $y(\bar{p})$ is signle valued, then $\frac{\partial \Pi(\bar{p})}{\partial p_i} = y_i(\bar{p})$

Proof: (i) is trivial.
(ii)Let $p'' = \alpha p + (1 - \alpha p'$, $\Pi(p'') = \alpha p y'' + (1 - \alpha) p y'' \leq \alpha \Pi(p) + (1 - \alpha)\Pi(p')$
(iii) is trivial
(iv) This is exactly the same as the proof for the Marshallian demand and Hicksian demand, and I will not repeat here. But I suppose you guys should practice at home. The last properties is a direct implication of envelope theorem.

4.3.2 Cost minimization

$$min_{z \geq 0} w.z s.t. f(z) \geq q$$

Denote the optimal solution as $z(w, q)$, conditional factor demand correspondence (function if single-valued) and $c(w, q)$ as the cost function.

Proposition 5.C.2: Assume Y is closed and satisfies the free disposal property. Then,
(i)$c(w, q)$ is homogenous of degree one in w and nondecreasing in q.
(ii)$c(w, q)$ is concave in w.
(iii) $z(w, q)$ is homogeneous of degree zero in w.
(iv) If $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w, q)$ is single-valued.
(v) (Shepard's lemma) $\frac{\partial c(w,q)}{\partial w_i} = z_i(w, q), \forall i$
(vi) f is homogeneous of degree 1 (CRS)$\Rightarrow$ $c(w, q)$ and $z(w, q)$ are homogenous of degree one in q. (vii) f is concave $\Rightarrow$ $c(w, q)$ is convex in q, in particular, marginal costs are nondecreasing in q.

Proof: (i) is trivial. Suppose $q \geq q'$, then any feasible plan with q is also feasible under q'. $c(w,q) = w.z \geq c(w,q')$.
(ii) $c(w'',q) = c(\alpha w + (1-\alpha)w', q) = \alpha w.z'' + (1-\alpha)w'.z'' \geq \alpha c(w,q) + (1-\alpha)c(w',q)$
(iii) Trivial
(iv) The same
(v) Follows the envelope theorem (vi) $c(w,\alpha q) = \min\{w.z,\ s.t. f(z) \geq \alpha q\} = \min\{w.z,\ s.t. f(z)/\alpha \geq q\} = \alpha \min\{w.(z/\alpha),\ s.t. f(z/\alpha) \geq q\} = \alpha c(w,q)$**.**
(vii) Note that w is fixed. Let z and z' be the solution under q and q'. Since f is concave, then

$$f(\alpha z + (1-alpha)z') \geq \alpha f(z) + (1-\alpha)f(z') \geq \alpha q + (1-\alpha)q'$$

Thus,

$$w.(\alpha z + (1-\alpha)z') \geq c(w, f(\alpha z + (1-\alpha)z')) \geq c(w, \alpha q + (1-\alpha)q')$$
$$\alpha c(w,q) + (1-\alpha)c(w,q') \geq c(w, \alpha q + (1-\alpha)q')$$

Given the cost function, we can rewrite the PMP as

$$\max_{q \geq 0} pq - c(w,q)$$

From the FOC: $p \leq \frac{\partial c(w,q)}{\partial q}$ with equality if $q > 0$, that is at the interior solution $\Leftrightarrow MR = MC$

Example: Suppose $f(z_1,z_2) = z_1^{\alpha} z_2^{\beta}$, it would be a good practice to solve the CMP.

4.4 The geometry of cost and supply in the single-output case

cost function as $C(q) = c(\bar{w}, q)$.

Average cost function: $AC(q) = C(q)/q$

Marginal cost function: $C'(q)$

Insert graph here:

5 Choice under Uncertainty

5.1 Expected utility theory

C: the set of all possible outcomes. Assume it is a finite set with N outcomes.

Definition: A simple lottery L is a list $L = (p_1, \cdots, p_N)$ **with** $p_n \geq 0, \forall n$ **and** $\sum_n p_n = 1$**, where** p_n **is the probability of outcome n occurring.**
For example: if we are talking about the real lottery, the outcomes are either 0 or 1 millian. The associated probability is then $(p_1, p_2) = (1 - 1/1m, 1/1m)$**.**

Insert graph here.

Definition: Given K simple lotteries, $L_k = (p_1^k, \cdots, p_N^K), k = 1, \cdots, K$**, and probabilities** $\alpha_k \geq 0$ **with** $\sum_k \alpha_k = 1$**, the compound lottery is** $(L_1, \cdots, L_K; \alpha_1, \cdots \alpha_K)$ **is the risky alternative that yields the simple lottery** L_k **with probability** $\alpha_k, \forall k$**.**

Reduced lottery of any compound lottery $(L_1, \cdots, L_K; \alpha_1, \cdots \alpha_K)$ **is** $L = (p_1, \cdots, p_N)$**, where** $p_n = \alpha_1 p_n^1 + \cdots + \alpha_K p_n^K$
For example: $L_1 = (0.9, 0.1)$, $L_2 = (0.1, 0.9)$ **are the simple lottery for outcomes winning a real car and winning a toy car. Then a compound lottery** $(L_1, L_2; 0.5, 0.5)$ **meaning with half half you will get either lottery. Therefore, the reduced form lottery for this compound lottery is** $L = (0.5 * 0.9 + 0.5 * 0.1, 0.5 * 0.1 + 0.5 * 0.9) = (0.5, 0.5)$.

5.2 Preference over Lotteries

$\mathfrak{L}$ **: the set of all simple lotteries over the set of outcomes** $\mathcal{C}$**. We assume that the preference is ration: completeness and transitivity.**

Definition: $\succcurlyeq$ **on** $\mathfrak{L}$ **is continuous if** $\forall L, L', L'' \in \mathfrak{L}$ **(the set of all simple lottery), the sets**

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha) L' \succcurlyeq L''\}$$

and

$$\{\alpha \in [0,1] : L'' \succcurlyeq \alpha L + (1-\alpha) L'\}$$

are closed.

Definition: $\succcurlyeq$ **on** $\mathfrak{L}$ **satisfies the independence axiom if** $\forall, L, L', L'', \alpha \in (0,1)$**,**

$$L \succcurlyeq L' \Leftrightarrow \alpha L + (1-\alpha)L'' \succcurlyeq \alpha L' + (1-\alpha)L''$$

$U(L) : \mathfrak{L} \to \mathbb{R}$ **is called a vonNeuman-MOrgenstern (v.N-M) expected utility function if** $\exists\ u_1, \cdots, u_N$**, such that**

$$U(L) = p_1 u_1 + \cdots + p_N u_N, \forall\ simple\ lottery L = (p_1, \cdots, p_N)$$

Proposition: If and only if $U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k), \forall L_k, \alpha_k$ **with** $\sum \alpha_k = 1$**, then the utility function is a v.N-M expected utility function.**
Proof: $\Rightarrow$ **let** $L^1 = (1, 0, \cdots, 0), L^2 = (0, 1, \cdots, 0), \cdots, L^N = (0, 0, \cdots, 1)$ **(degenerate lotteries), then**

$$U(L) = U(\sum p_n L^n) = \sum p_n U(L^n)$$

$\Leftarrow$**,**

$$U(L_k) = \sum_{n=1}^{N} p_n^k u^n$$

$$U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_n \left(u_n \sum_k \alpha_k p_n^k \right) = \sum_k \alpha_k \left(\sum_n u_n p_n^k \right) = \sum_k \alpha_k U(L_k)$$

Proposition: If $U(L)$ **is a v.N-M expected utility function, then** $\tilde{U}(L)$ **is another one iff** $\exists \beta > 0\ and\ \gamma$ **such that** $\tilde{U}(L) = \beta U(L) + \gamma, \forall L \in \mathfrak{L}$
$\Rightarrow$**,**

$$\begin{aligned} \tilde{U}(\sum_k \alpha_k L_k) &= \beta U(\sum_k \alpha_k L_k) + \gamma \\ &= \beta \left[\sum_k \alpha_k U(L_k) \right] + \gamma \\ &= \sum_k \alpha_k [\beta U(L_k) + \gamma] \\ &= \sum_k \alpha_k \tilde{U}(L_k) \end{aligned}$$

$\Leftarrow$ **Pick** $\underline{L}$ **and** $\overline{L}$ **such that** $\overline{\succcurlyeq} L \succcurlyeq \underline{L}, \forall L$**. If** $\overline{L} \sim \underline{L}$**, then every utility function is a constant and the result follows immediately. Thus, we assume** $\overline{L} \succ \underline{L}$**.** $\forall L$ **and define** $\lambda_L \in [0, 1]$

by $U(L) = \lambda_L U(\overline{L}) + (1-\lambda_L)U(\underline{L})$. Thus,

$$\lambda_L = \frac{U(L) - U(\underline{L})}{U(\overline{L} - U(\underline{L}))}$$

Since $\lambda_L U(\overline{L}) + (1-\lambda_L)U(\underline{L}) = U(\lambda_L \overline{L} + (1-\lambda_L)\underline{L})$ and U represents $\succcurlyeq$, it must be that $L \sim \lambda \overline{L} + (1-\lambda_L)\underline{L}$.Since $\tilde{U}$ also represents the same preference, we have

$$\begin{aligned}
\tilde{U}(L) &= \tilde{U}(\lambda_L \overline{L} + (1-\lambda_L)\underline{L}) \\
&= \lambda_L \tilde{U}(\overline{L}) + (1-\lambda_L)\tilde{U}(\underline{L}) \\
&= \frac{\tilde{U}(\overline{L}) - \tilde{U}(\underline{L})}{U(\overline{L}) - U(\underline{L})} U(L) + \frac{\tilde{U}(\underline{L}) - \tilde{U}(\overline{L})}{U(\overline{L}) - U(\underline{L})} . U(\underline{L})
\end{aligned}$$

If $\succcurlyeq$ on the space of lotteries $\mathfrak{L}$ is rational, continuous and satisfies the independence axiom, then $\succcurlyeq$ can be represented by an expected utility function. That is,

$$L \succcurlyeq L' \Leftrightarrow \sum_n u_n p_n \geq \sum_n u_n p'_n, \forall L, L' \in \mathfrak{L}$$

Proof: Assume that there are best and worst lotteries with $\overline{L} \succ \underline{L}$.
Step 1. If $L \succ L'$ and $\alpha \in [0,1)$, then $L \succ \alpha L + (1-\alpha)L'$; If $L \succ L'$ and $\alpha \in (0,1]$, then $\alpha L + (1-\alpha)L' \succ L'$, this is because for $\alpha \in (0,1)$,

$$L = \alpha L + (1-\alpha)L \succ \alpha L + (1-\alpha)L' \succ \alpha L' + (1-\alpha)L' = L'$$

. For $\alpha = 0$ *or* 1, it follows directly.
Step 2: Let $\alpha, \beta \in [0,1]$. Then $\beta \overline{L} + (1-\beta)\underline{L} \succ \alpha \overline{L} + (1-\alpha)\underline{L}$ iff $\beta > \alpha$
$\Leftarrow$: If $\beta > \alpha$, this implies $\alpha < 1$.

$$\beta \overline{L} + (1-\beta)\underline{L} = \gamma \overline{L} + (1-\gamma)[\alpha \overline{L} + (1-\alpha)\underline{L}],$$

where $\gamma = (\beta - \alpha)/(1-\alpha) \in (0,1]$. By step 1: $\overline{L} \succ \alpha \overline{L} + (1-\alpha)\underline{L}$. Applying step 1 again: $\gamma \overline{L} + (1-\gamma)(\alpha \overline{L} + (1-\alpha)\underline{L}) \succ \alpha \overline{L} + (1-\alpha)\underline{L}$
$\Rightarrow$, using contradiction. Suppose $\beta \geq \alpha$. If $\beta = \alpha$, then $\beta \overline{L} + (1-\beta)\underline{L} \sim \alpha \overline{L} + (1-\alpha)\underline{L}$. Contradiction directly. If $\beta > \alpha$, then from the prove above we have $\alpha \overline{L} + (1-\alpha)\underline{L} \succ \beta \overline{L} + (1-\beta)\underline{L}$. Contradiction.
Step 3: For any $L \in \mathfrak{L}$, there is a unique α_L such that $[\alpha \overline{L} + (1-\alpha)\underline{L}] \sim L$. The existence is quite similar to that used for the one with certainty. The main argument is from the continuity of the preference. We will not repeat here. The uniqueness follows from step 2.
Step 4: The function $U : \mathfrak{L} \to \mathbb{R}$ with $U(L) = \alpha_L$ represents $\succcurlyeq$.
$L \succcurlyeq L'$ iff $\alpha_L \overline{L} + (1-\alpha_L)\underline{L} \succcurlyeq \alpha_{L'} \overline{L} + (1-\alpha_{L'})\underline{L}$ by step 3. Thus, $L \succcurlyeq L'$ iff $\alpha_L \geq \alpha_{L'}$
Step 5: The utility function $U(\cdot)$ is linear and therefore has the expected utility form.

We need to show that $U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta)U(L')$**.**

$$L \sim U(L)\overline{L} + (1 - U(L))\underline{L}$$

and

$$L' \sim U(L')\overline{L} + (1 - U(L'))\underline{L}$$

Therefore, by the independence axiom,

$$\begin{aligned} \beta L + (1-\beta)L' &\sim \beta[U(L)\overline{L} + (1 - U(L)\underline{L})] + (1-\beta)L' && (5) \\ &\sim \beta[U(L)\overline{L} + (1 - U(L)\underline{L})] + (1-\beta)[U(L')\overline{L} + (1 - U(L')\underline{L})] && (6) \\ &= [\beta U(L) + (1-\beta)U(L')]\overline{L} + [1 - \beta U(L) - (1-\beta)U(L')]\underline{L} && (7) \end{aligned}$$

Thus, $U(\beta L + (1-\beta)) = \beta U(L) + (1-\beta)U(L')$**.**

Here is a graphic proof for the case with three outcomes. Insert graph here.

5.3 Allais Paradox

$$\begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \quad (8)$$

$$\begin{array}{ccc} outcome\ 1 & outcome\ 1 & outcome\ 3 \end{array} \quad (9)$$

$$\begin{array}{ccc} 2.5m & 0.5m & 0 \end{array} \quad (10)$$

Choose $L_1 = (0, 1, 0), L_1' = (0.1, 0.89, 0.01)$**, you should choose** L_1**.**
Choose $L_2 = (0, 0.11, 0.89), L_2' = (0.1, 0, 0.9)$**, you should choose** L_2'**.**

$$L_1 \succ L_1' \Rightarrow u_2 > 0.1u_1 + 0.89u_2 + 0.01u_3 \Rightarrow 0.11u_2 > 0.1u_1 + 0.01u_3 \quad (11)$$

$$L_2' \succ L_2 \Rightarrow 0.11u_2 + 0.89u_3 < 0.10u_1 + 0.9u_3 \Rightarrow 0.11u_2 < 0.1u_1 + 0.01u_3 \quad (12)$$

Contradiction. Insert graph here.

A preference ordering satisfies weak independence if: $\forall L \sim L' \in \mathfrak{L} \Rightarrow \forall \theta \in (0,1), \exists \beta \in (0,1)$ **such that** $\forall L'' \in \mathfrak{L}$**,** $\theta L + (1-\theta)L'' \sim \beta L' + (1-\beta)L''$

Insert graph here.

$\forall L, L', L'' \in \mathfrak{L}, L \sim L' \Rightarrow \forall \theta \in (0,1), \exists \beta \in (0,1)$ **such that** $\theta L + (1-\theta)L'' \sim \beta L' + (1-\beta)L''$**.**

$$U(L) = \frac{\sum p_i w_i(U(L)) u_i}{\sum p_i w_i(U(L))}$$

Insert graph here.

The preference satisfies betweenness if: $\forall L, L'$ **and** $\lambda \in (0,1)$**,** $L \sim L' \Rightarrow \lambda L + (1-\lambda)L' \sim L$

This implies linear indifference curves and the utility function takes the form

$$U(L) = \sum u_i(U(p))p_i$$

Insert graph here.

5.4 Money lotteries and Risk Aversion

Outcomes: $X = (-\infty, +\infty)$
The lottery now is a distribution defined on the outcome $X = (-\infty, +\infty)$**. Denote cdf F(x) and pdf f(x)**

Definition: A person is risk averse if the person prefers $\int xf(x)dx$ **to** $F(x), \forall F(\cdot)$**; is risk neutral if indifferent** $\forall F(\cdot)$**; is strictly risk averse if indifferent only when the two lotteries are the same, i.e** $F(\cdot)$ **is degenerate.**
Definition: The Bernoulli utility is concave if $\int u(x)dF(x) \leq u(\int xdF(x)), \forall F(\cdot)$**. This is also called the Jensen's inequality.**

Definition: A person is risk lover, if he prefers $F(x)$ **to** $\int xdF(x)$**, which is equivalent to** $u(\cdot)$ **being convex.** In real life, a person is risk averse on some range and risk lover on other ranges. That is why we buy insurance, however, we play lottery as well.

Definition: The certainty equivalent of $F(\cdot)$**,** $c(F, u)$**, is the certain amount such that** $u(c(F, u)) = \int u(x)dF(x)$
The probability premium $\pi(x, \epsilon, u)$ **is the excess in winning probability over fair odds that makes the individual indfferent between the certain outcome** x **and a gamble between the two outcomes** $x+\epsilon$ **and** $x-\epsilon$**:** $u(x) = (0.5+\pi(x, \epsilon, u))u(x+\epsilon)+(0.5-\pi(x, \epsilon, u))u(x-\epsilon)$
Graphically, this can be shown Insert graph here.

Proposition: The following properties are equivalent:
(i) The decision maker is risk averse
(ii) $u(\cdot)$ **is concave**
(iii) $c(F,u) \leq \int x dF(x), \forall F(\cdot)$
(iv) $\pi(x,\epsilon,u) \geq 0, \forall x, \epsilon$
Proof: $(i) \Leftrightarrow (ii)$**, since the decision maker is risk averse, he prefers** $\int x dF(x)$ **to** $F(x)$**, i.e.** $U(\int x dF(x)) \geq U(F(x))$**. Furthermore, since the preference can be represented by expected utility, this is equivalent to** $\int x dF(x) \geq \int u(x) dF(x)$**, which is exactly the definition for** $u(\cdot)$ **being concave**
$(ii) \Leftrightarrow (iii)$: $\int u(x)dF(x) \leq u(\int x dF(x)) \Leftrightarrow u(c(F,u)) \leq u(\int x dF(x)) \Leftrightarrow c(F,u) \leq \int x dF(x)$**. The first step follows the definition of** $c(F,u)$**, and the second step follows** $u(\cdot)$ **being increasing. For (iv), I will not prove it here and leave it as an exercise.**

Example: Insurance, a strictly risk aversion person with initial wealth w
Car accidents: occurs with prob π **and loss** D
A firm offers insurance: premium $q = \pi$ **for one unit of insurance (pay you 1 dollar if the accident occurs). This is called actuarially fair in the sense that the insurance company earns zero profit.**
Then the person's problem is

$$\max_{\alpha \geq 0} (1-\pi)u(w - \alpha\pi) + \pi u(w - \alpha\pi + \alpha - D)$$

The FOC yields:

$$-u'(w - \alpha\pi) + u'(w - \alpha\pi + \alpha - D) \leq 0$$

with equality if $\alpha > 0$**. It is easy to check that the corner solution** $\alpha = 0$ **is not optimal. Thus, it must be interior solution. Since u is strictly concave, it must be** $\alpha = D$**.**

Example: Investment: invest in a bank or stock market
Bank: interest r———α
Stock market: interest z **follows** $F(z)$———β
Assume $E(z) > r$
Total wealth: $w = \alpha + \beta$
Now the person's problem is

$$\max_{\alpha,\beta \geq 0} \int u(\alpha z + \beta r + w) dF(z)$$

st. $\alpha + \beta = w$ **Thus,**

$$\max_{0 \le \alpha \le w} \int u(\alpha z + r(w - \alpha) + w) dF(z) = \int u(\alpha(z - r) + w(1 + r)) dF(z)$$

$$L = \int u(\alpha(z - r) + w(1 + r)) dF(z) + \lambda_1 \alpha + \lambda_2 (w - \alpha)$$

The Kuhn-Tucker FOC yields

$$\int u'(\alpha(z - r) + w(1 + r))(z - r) dF(z) + \lambda_1 - \lambda_2 = 0$$

$$\lambda_1 \alpha = 0, \alpha \ge 0, \lambda_1 \ge 0$$

$$\lambda_2 (w - \alpha) = 0, w - \alpha \ge 0, \alpha_2 \ge 0$$

First consider the corner solution $\alpha = 0$**, then** $\lambda_2 = 0$**. Thus,**

$$\int u'(w(1 + r))(z - r) dF(z) + \lambda_1 > 0$$

Thus $\alpha > 0$**. This means that he will invest with positive amount for sure. The conclusion is that if a risk is actuarially favorable, then a risk averter will always accept at least small amount of it.**

Measurement of Risk Aversion

Definition: Arrow-Pratt coefficient of absolute risk aversion $r_A(x) = -\frac{u''(x)}{u'(x)}$
The straighforward question is why this can measure the size of risk aversion. Insert graph here.

Yet, another commonly used measurement is called the relative risk aversion

$$r_R(x) = -\frac{x u''(x)}{u(x)}$$

Note that decreasing relative risk aversion implies decreasing absolute risk aversion.

Comparisons across individuals

We say u2 is more risk averse than u1 if
(i) $r_A(x, u_2) \geq r_A(x, u), \forall x$
(ii)$r_R(x, u_2) \geq r_R(x, u_1), \forall x$
(ii)$\exists$ **increasing concave function, such that** $u_2(x) = \varphi, (u_1(x), \forall x$
(iii)$C(F, u_2) \leq C(F, u_1), \forall F(\cdot)$
(iv)$\pi(x, \epsilon, u_2) \geq \pi(x, \epsilon, u_1), \forall x, \epsilon$
(v)$\int u_2(x)dF(x) \geq u_2(\bar{x}) \Rightarrow \int u_1(x)dF(x) \geq u_1(\bar{x}), \forall F(\cdot), \bar{x}$

5.5 Comparing return and risk

Example, you have the following lotteries:

	\$100	\$200
L_1	1/3	2/3
L_2	2/3	1/3

	\$100	\$200	\$300
L_1	1/10	6/10	3/10
L_2	2/10	7/10	1/10

Definition: $F(x)$ **first-order stochastic dominates** $G(x)$ **if**

$$\int u(x)dF(x) \geq \int u(x)dG(x), \forall increasing\ function\ u(\cdot)$$

which is equivalent to $F(x) \leq G(x), \forall x$**.**

Proof:

$$\int u(x)dF(x) \geq \int u(x)dG(x) \tag{13}$$

$$\Leftrightarrow u(x)F(x)|_{x=-\infty}^{\infty} - \int F(x)u'(x)dx \geq u(x)G(x)|_{x=-\infty}^{\infty} - \int G(x)u'(x)dx \tag{14}$$

$$\Leftrightarrow \int u'(x)(F(x) - G(x)) \geq 0 \tag{15}$$

"$\Leftarrow$" **follows directly.** "$\Rightarrow$"**: suppose not, and** $F(x) < G(x)$ **on certain interval, then we can let** u **be strictly increasing on this interval and constant elsewhere. Then** $\int u'(x)(F(x) - G(x)) < 0$**, a contradiction.**

Insert graph here.

Definition: $F(x)$ **second-order stochastically dominates** $G(x)$ **if**

$$\int u(x)dF(x) \geq \int u(x)dG(x), \forall\ increasing\ concave\ function\ u$$

Example: Mean-preserving spreads. Consider the compound lottery $x+z$**, where both x and z are both simple lotteries, with the mean of z to be zero. x and z may or maynot be independent. Denote the cdf of x as** $F(x)$ **and cdf of z as** $H_x(z)$**. Let the reduced form of the compound lottery denote as y with cdf** $G(y)$**. In this case, we say** $G(\cdot)$ **is a mean-preserving spread of** $F(\cdot)$**. We can show that with the same mean, G is a mean-preserving spread of F(.) implies F second order stochastic dominant G.**

$$\begin{aligned} \int u(x)dG(x) &= \int (\int u(x+z)dH_x(z)dF(x)) && (16) \\ &\leq \int u(\int (x+z)dH_x(z))dF(x) && (17) \\ &= \int u(x)dF(x) && (18) \end{aligned}$$

Alternative definition for second order stochastic dominance
Consider F **and** G **have the same mean.**

$$0 = \int xdF - \int xdG = \int xd(F-G) = x(F-G)|_{x=-\infty}^{\infty} - \int (F-G)dx \tag{19}$$

Thus, $\int [F(x) - G(x)]dx = 0$ Insert graph here.

$$\int_0^x G(t)dt \geq \int_0^x F(t)dt, \forall x$$

It turns out that this definition is equivalent to F second order stochastically dominating G.

So our conclusion is that with the same mean, second order stochastic dominance is equivalent to mean preserving spread and is equivalent to the above condition.

6 Game Theory

5.2 Description of a game

There are several ways of describing a game. For our purposes, the strategic form and the extensive form will be sufficient. Roughly speaking the extensive form provides an "extended" description of a game while the strategic form provides a "reduced" summary of a game. We will first describe the strategic form, reserving the discussion of the extensive form for the section on sequential games.

5.2.1 Strategic Form

The strategic form of the game is defined by exhibiting a set of players $N = \{1, 2, \ldots, n\}$. Each player i has a set of strategies S_i from which he/she can choose an action $s_i \in S_i$ and a payoff function, $\phi_i(s)$, that indicate the utility that each player receives if a particular combination s of strategies is chosen, where $s = (s_1, s_2, \ldots, s_n) \in S = \prod_{i=1}^{n} S_i$. For purposes of exposition, we will treat two-person games in this chapter. All of the concepts described below can be easily extended to multi-person contexts.

We assume that the description of the game – the payoffs and the strategies available to the players – are common knowledge. That is, each player knows his own payoffs and strategies, and the other player's payoffs and strategies. Furthermore, each player knows that the other player knows this, and so on. We also assume that it is common knowledge that each player is "fully rational." That is, each player can choose an action that maximizes his utility given his subjective beliefs, and that those beliefs are modified when new information arrives according to Bayes' law.

Game theory, by this account, is a generalization of standard, one-person decision theory. How should a rational expected utility maximizer behave in a situation in which his payoff depends on the choices of another rational expected utility maximizer? Obviously, each player will have to consider the problem faced by the other player in order to make a sensible choice. We examine the outcome of this sort of consideration below.

Example 5.2.1 (Matching pennies) In this game, there are two players, Row and Column. Each player has a coin which he can arrange so that either the head side or the tail side is face-up. Thus, each player has two strategies which we abbreviate as Heads or Tails. Once the strategies are chosen there are payoffs to each player which depend on the choices that both players make.

These choices are made independently, and neither player knows the other's choice when he makes his own choice. We suppose that if both players show heads or both show tails, then Row wins a dollar and Column loses a dollar. If, on the other hand, one player exhibits heads and the other exhibits tails, then Column wins a dollar and Row looses a dollar.

		Column	
		Heads	Tails
Row	Heads	$(1,-1)$	$(-1,1)$
	Tails	$(-1,1)$	$(1,-1)$

Table 5.1: Game Matrix of Matching Pennies

We can depict the strategic interactions in a game matrix. The entry in box (Head, Tails) indicates that player Row gets -1 and player Column gets $+1$ if this particular combination of strategies is chosen. Note that in each entry of this box, the payoff to player Row is just the negative of the payoff to player Column. In other words, this is a zero-sum game. In zero-sum games the interests of the players are diametrically opposed and are particularly simple to analyze. However, most games of interest to economists are not zero sum games.

Example 5.2.2 (The Prisoners Dilemma) Again we have two players, Row and Column, but now their interests are only partially in conflict. There are two strategies: to Cooperate or to Defect.

In the original story, Row and Column were two prisoners who jointly participated in a crime. They could cooperate with each other and refuse to give evidence (i.e., do not confess), or one could defect (i.e, confess) and implicate the other. They are held in separate cells, and each is privately told that if he is the only one to confess, then he will be rewarded with a light sentence of 1 year while the recalcitrant prisoner will go to jail for 10 years. However, if the person is the only one not to confess, then it is the who will serve the 10-year sentence. If both confess, they will both be shown some mercy: they will each get 5 years. Finally, if neither confesses, it will still possible to convict both of a lesser crime that carries a sentence of 2 years. Each player wishes to minimize the time he spends in jail. The outcome can be shown in Table 5.2.

		Prisoner 2	
		Don't Confess	Confess
Prisoner 1	Don't Confess	$(-2, -2)$	$(-10, -1)$
	Don't confess	$(-1, -10)$	$(-5, -5)$

Table 5.2: The Prisoner's Dilemma

The problem is that each party has an incentive to confess, regardless of what he or she believes the other party will do. In this prisoner's dilemma, "confession" is the best strategy to each prisoner regardless the choice of the other.

An especially simple revised version of the above prisoner's dilemma given by Aumann (1987) is the game in which each player can simply announce to a referee: "Give me $\$1,000$," or "Give the other player $\$3,000$." Note that the monetary payments come from a third party, not from either of the players; the Prisoner's Dilemma is a variable-sum game.

The players can discuss the game in advance but the actual decisions must be independent. The Cooperate strategy is for each person to announce the $\$3,000$ gift, while the Defect strategy is to take the $\$1,000$ (and run!). Table 5.3 depicts the payoff matrix to the Aumann version of the Prisoner's Dilemma, where the units of the payoff are thousands of dollars.

		Column	
		Cooperate	Defect
Row	Cooperate	$(3, 3)$	$(0, 4)$
	Defect	$(4, 0)$	$(1, 1)$

Table 5.3: A Revised Version of Prisoner's Dilemma by Aumann

We will discuss this game in more detail below. Again, each party has an incentive to defect, regardless of what he or she believes the other party will do. For if I believe that the other person will cooperate and give me a $\$3,000$ gift, then I will get $\$4,000$ in total by defecting. On the other hand, if I believe that the other person will defect and just take the $\$1,000$, then I do better by taking $\$1,000$ for myself.

In other applications, cooperate and defect could have different meanings. For example, in a duopoly situation, cooperate could mean "keep charging a high price" and defect could mean "cut your price and steal your competitor's market."

Example 5.2.3 (Cournot Duopoly) Consider a simple duopoly game, first analyzed by Cournot (1838). We suppose that there are two firms who produce an identical good with a marginal cost c. Each firm must decide how much output to produce without knowing the production decision of the other duopolist. If the firms produce a total of x units of the good, the market price will be $p(x)$; that is, $p(x)$ is the inverse demand curve facing these two producers.

If x_i is the production level of firm i, the market price will then be $p(x_1+x_2)$, and the profits of firm i are given by $\pi_i(p(x_1+x_2)-c)x_i$. In this game the strategy of firm i is its choice of production level and the payoff to firm i is its profits.

Example 5.2.4 (Bertrand duopoly) Consider the same setup as in the Cournot game, but now suppose that the strategy of each player is to announce the price at which he would be willing to supply an arbitrary amount of the good in question. In this case the payoff function takes a radically different form. It is plausible to suppose that the consumers will only purchase from the firm with the lowest price, and that they will split evenly between the two firms if they charge the same price. Letting $x(p)$ represent the market demand function and c the marginal cost, this leads to a payoff to firm 1 of the form:

$$\pi_1(p_1,p_2) = \begin{cases} (p_1-c)x(p_1) & if \quad p_1 < p_2 \\ (p_1-c)x(p_1)/2 & if \quad p_1 = p_2 \\ 0 & if \quad p_1 > p_2 \end{cases}.$$

This game has a similar structure to that of the Prisoner's Dilemma. If both players cooperate, they can charge the monopoly price and each reap half of the monopoly profits. But the temptation is always there for one player to cut its price slightly and thereby capture the entire market for itself. But if both players cut price, then they are both worse off.

Note that the Cournot game and the Bertrand game have a radically different structure, even though they purport to model the same economic phenomena – a duopoly. In the Cournot game, the payoff to each firm is a continuous function of its strategic choice; in the Bertrand game, the payoffs are discontinuous functions of the strategies. As might be expected, this leads to quite different equilibria. Which of these models is reasonable? The answer is that it depends on what you are trying to model. In most economic modelling, there is an art to choosing a representation of the strategy choices of the game that captures an element of the real strategic iterations, while at the same time leaving the game simple enough to analyze.

5.3 Solution Concepts

5.3.1 Mixed Strategies and Pure Strategies

In many games the nature of the strategic interaction suggests that a player wants to choose a strategy that is not predictable in advance by the other player. Consider, for example, the Matching Pennies game described above. Here it is clear that neither player wants the other player to be able to predict his choice accurately. Thus, it is natural to consider a random strategy of playing heads with some probability p_h and tails with some probability p_t. Such a strategy is called a **mixed strategy**. Strategies in which some choice is made with probability 1 are called **pure strategies**.

If R is the set of pure strategies available to Row, the set of mixed strategies open to Row will be the set of all probability distributions over R, where the probability of playing strategy r in R is p_r. Similarly, p_c, will be the probability that Column plays some strategy c. In order to solve the game, we want to find a set of mixed strategies (p_r, p_c) that are, in some sense, in equilibrium. It may be that some of the equilibrium mixed strategies assign probability 1 to some choices, in which case they are interpreted as pure strategies.

The natural starting point in a search for a solution concept is standard decision theory: we assume that each player has some probability beliefs about the strategies that the other player might choose and that each player chooses the strategy that maximizes his expected payoff.

Suppose for example that the payoff to Row is $u_r(r, c)$ if Row plays r and Column plays c. We assume that Row has a subjective probability distribution over Column's choices which we denote by (π_c); see Chapter 4 for the fundamentals of the idea of subjective probability. Here π_c is supposed to indicate the probability, as envisioned by Row, that Column will make the choice c. Similarly, Column has some beliefs about Row's behavior that we can denote by (π_c).

We allow each player to play a mixed strategy and denote Row's actual mixed strategy by (p_r) and Column's actual mixed strategy by (p_c). Since Row makes his choice without knowing Column's choice, Row's probability that a particular outcome (r, c) will occur is $p_r\pi_c$. This is simply the (objective) probability that Row plays r times Row's (subjective) probability that Column plays c. Hence, Row's objective is to choose a probability distribution (p_r) that maximizes

Row's expected payoff $= \sum_r \sum_c p_r \pi_c u_r(r, c)$.

Column, on the other hand, wishes to maximize

Column's expected payoff $= \sum_c \sum_r p_c \pi_r u_c(r, c)$.

So far we have simply applied a standard decision-theoretic model to this game – each player wants to maximize his or her expected utility given his or her beliefs. Given my beliefs about what the other player might do, I choose the strategy that maximizes my expected utility.

5.3.2 Nash equilibrium

In the expected payoff formulas given at the end of the last subsection, Row's behavior — how likely he is to play each of his strategies represented by the probability distribution (p_r) and Column's beliefs about Row's behavior are represented by the (subjective) probability distribution (π_r).

A natural consistency requirement is that each player's belief about the other player's choices coincides with the actual choices the other player intends to make. Expectations that are consistent with actual frequencies are sometimes called rational expectations. A Nash equilibrium is a certain kind of rational expectations equilibrium. More formally:

Definition 5.3.1 (Nash Equilibrium in Mixed Strategies.) A Nash equilibrium in mixed strategies consists of probability beliefs (π_r, π_c) over strategies, and probability of choosing strategies (p_r, p_c), such that:

1. the beliefs are correct: $p_r = \pi_r$ and $p_c = \pi_c$ for all r and c; and,
2. each player is choosing (p_r) and (p_c) so as to maximize his expected utility given his beliefs.

In this definition a Nash equilibrium in mixed strategies is an equilibrium in actions and beliefs. In equilibrium each player correctly foresees how likely the other player is to make various choices, and the beliefs of the two players are mutually consistent.

A more conventional definition of a Nash equilibrium in mixed strategies is that it is a pair of mixed strategies (p_r, p_c) such that each agent's choice maximizes his expected utility, given the strategy of the other agent. This is equivalent to the definition we use, but it is misleading since the distinction between the beliefs of the agents and the actions of the agents is blurred. We've tried to be very careful in distinguishing these two concepts.

One particularly interesting special case of a Nash equilibrium in mixed strategies is a Nash equilibrium in pure strategies, which is simply a Nash equilibrium in mixed strategies in which the probability of playing a particular strategy is 1 for each player. That is:

Definition 5.3.2 (Nash equilibrium in Pure Strategies.) A Nash equilibrium in pure strategies is a pair (r^*, c^*) such that $u_r(r^*, c^*) \geqq u_r(r, c^*)$ for all Row strategies r, and $u_c(r^*, c^*) \geqq u_c(r^*, c)$ for all Column strategies c.

A Nash equilibrium is a minimal consistency requirement to put on a pair of strategies: if Row believes that Column will play c^*, then Row's best reply is r^* and similarly for Column. No player would find it in his or her interest to deviate unilaterally from a Nash equilibrium strategy.

If a set of strategies is not a Nash equilibrium then at least one player is not consistently thinking through the behavior of the other player. That is, one of the players must expect the other player not to act in his own self-interest – contradicting the original hypothesis of the analysis.

An equilibrium concept is often thought of as a "rest point" of some adjustment process. One interpretation of Nash equilibrium is that it is the adjustment process of "thinking through" the incentives of the other player. Row might think: "If I think that Column is going to play some strategy c_1 then the best response for me is to play r_1. But if Column thinks that I will play r_1, then the best thing for him to do is to play some other strategy c_2. But if Column is going to play c_2, then my best response is to play r_2..." and so on.

Example 5.3.1 (Nash equilibrium of Battle of the Sexes) The following game is known as the "Battle of the Sexes." The story behind the game goes something like this. Rhonda Row and Calvin Column are discussing whether to take microeconomics or macroeconomics this semester. Rhonda gets utility 2 and Calvin gets utility 1 if they both take micro; the payoffs are reversed if they both take macro. If they take different courses, they both get utility 0.

Let us calculate all the Nash equilibria of this game. First, we look for the Nash equilibria in pure strategies. This simply involves a systematic examination of the best responses to various strategy choices. Suppose that Column thinks that Row will play Top. Column gets 1 from playing Left and 0 from playing Right, so Left is Column's best response to Row playing Top. On the other hand, if Column plays Left, then it is easy to see that it is optimal for Row to play Top. This line of reasoning shows that (Top, Left) is a Nash equilibrium. A similar argument shows that (Bottom, Right) is a Nash equilibrium.

		Calvin	
		Left (micro)	Right (macro)
Rhonda	Top (micro)	$(2, 1)$	$(0, 0)$
	Bottom (macro)	$(0, 0)$	$(1, 2)$

Table 5.4: Battle of the Sexes

We can also solve this game systematically by writing the maximization problem that each agent has to solve and examining the first-order conditions. Let (p_t, p_b) be the probabilities with which Row plays Top and Bottom, and define (p_l, p_r) in a similar manner. Then Row's problem is

$$\max_{(p_t,p_b)} p_t[p_l 2 + p_r 0] + p_b[p_l 0 + p_r 1]$$

such that

$$\begin{aligned} p_t &\geqq 0 \\ p_b &\geqq 0. \end{aligned}$$

Let λ, μ_t, and μ_b be the Kuhn-Tucker multipliers on the constraints, so that the Lagrangian takes the form:

$$\mathcal{L} = 2p_t p_l + p_b p_r - \lambda(p_t + p_b - 1) - \mu_t p_t - \mu_b p_b.$$

Differentiating with respect to p_t and p_b, we see that the Kuhn-Tucker conditions for Row are

$$\begin{aligned} 2p_l &= \lambda + \mu_t \\ p_r &= \lambda + \mu_b \end{aligned} \tag{5.1}$$

Since we already know the pure strategy solutions, we only consider the case where $p_t > 0$ and $p_b > 0$. The complementary slackness conditions then imply that , $\mu_t = \mu_b = 0$. Using the fact that $p_l + p_r = 1$, we easily see that Row will find it optimal to play a mixed strategy when $p_l = 1/3$ and $p_r = 2/3$.

Following the same procedure for Column, we find that $p_t = 2/3$ and $p_b = 1/3$. The expected payoff to each player from this mixed strategy can be easily computed by plugging these numbers into the objective function. In this case the expected payoff is 2/3 to each player. Note that each player would prefer either of the pure strategy equilibria to the mixed strategy since the payoffs are higher for each player.

Remark 5.3.1 One disadvantage of the notion of a mixed strategy is that it is sometimes difficult to give a behavioral interpretation to the idea of a mixed strategy although a mixed strategies are the only sensible equilibrium for some games such as Matching Pennies. For example, a duopoly game – mixed strategies seem unrealistic.

5.3.3 Dominant strategies

Let r_1 and r_2 be two of Row's strategies. We say that r_1 strictly dominates r_2 for Row if the payoff from strategy r_1 is strictly larger than the payoff for r_2 no matter what choice Column makes. The strategy r_1 weakly dominates r_2 if the payoff from r_1 is at least as large for all choices Column might make and strictly larger for some choice.

A **dominant strategy equilibrium** is a choice of strategies by each player such that each strategy (weakly) dominates every other strategy available to that player. One particularly interesting game that has a dominant strategy equilibrium is the Prisoner's Dilemma in which the dominant strategy equilibrium is (confess, confess). If I believe that the other agent will not confess, then it is to my advantage to confess; and if I believe that the other agent will confess, it is still to my advantage to confess.

Clearly, a dominant strategy equilibrium is a Nash equilibrium, but not all Nash equilibria are dominant strategy equilibria. A dominant strategy equilibrium, should one exist, is an especially compelling solution to the game, since there is a unique optimal choice for each player.

5.4 Repeated games

In many cases, it is not appropriate to expect that the outcome of a repeated game with the same players as simply being a repetition of the one-shot game. This is because the strategy space of the repeated game is much larger: each player can determine his or her choice at some point as a function of the entire history of the game up until that point. Since my opponent can modify his behavior based on my history of choices, I must take this influence into account when making my own choices.

Let us analyze this in the context of the simple Prisoner's Dilemma game described earlier. Here it is in the "long-run" interest of both players to try to get to the (Cooperate, Cooperate) solution. So it might be sensible for one player to try to "signal" to the other that he is willing to "be nice" and play cooperate on the first move of the game. It is in the short-run interest of the other player to Defect, of course, but is this really in his long-run interest? He might reason that if he defects, the other player may lose patience and simply play Defect himself from then on. Thus, the second player might lose in the will produce the Cournot output forever. An argument similar to the Prisoner's Dilemma argument shows that this is a Nash equilibrium.

5.5 Refinements of Nash equilibrium

The Nash equilibrium concept seems like a reasonable definition of an equilibrium of a game. As with any equilibrium concept, there are two questions of immediate interest: 1) will a Nash equilibrium generally exist; and 2) will the Nash equilibrium be unique?

Existence, luckily, is not a problem. Nash (1950) showed that with a finite number of agents and a finite number of pure strategies, an equilibrium will always exist. It may, of course, be an equilibrium involving mixed strategies. We will shown in Chapter 7 that it always exists a pure strategy Nash equilibrium if the strategy space is a compact and convex set and payoffs functions are continuous and quasi-concave.

Uniqueness, however, is very unlikely to occur in general. We have already seen that there may be several Nash equilibria to a game. Game theorists have invested a substantial amount of effort into discovering further criteria that can be used to choose among Nash equilibria. These criteria are known as refinements of the concept of Nash equilibrium, and we will investigate a few of them below.

5.5.1 Elimination of dominated strategies

When there is no dominant strategy equilibrium, we have to resort to the idea of a Nash equilibrium. But typically there will be more than one Nash equilibrium. Our problem then is to try to eliminate some of the Nash equilibria as being "unreasonable."

One sensible belief to have about players' behavior is that it would be unreasonable for them to play strategies that are dominated by other strategies. This suggests that when given a game, we should first eliminate all strategies that are dominated and then calculate the Nash equilibria of the remaining game. This procedure is called elimination of dominated strategies; it can sometimes result in a significant reduction in the number of Nash equilibria.

For example consider the game given in Table 5.5. Note that there are two pure strategy Nash equilibria, (Top, Left) and (Bottom, Right). However, the strategy Right

		Column	
		Left	Right
Row	Top	$(2, 2)$	$(0, 2)$
	Bottom	$(2, 0)$	$(1, 1)$

Table 5.5: A Game With Dominated Strategies

weakly dominates the strategy Left for the Column player. If the Row agent assumes that Column will never play his dominated strategy, the only equilibrium for the game is (Bottom, Right).

Elimination of strictly dominated strategies is generally agreed to be an acceptable procedure to simplify the analysis of a game. Elimination of weakly dominated strategies is more problematic; there are examples in which eliminating weakly dominated strategies appears to change the strategic nature of the game in a significant way.

5.5.2 Sequential Games and Subgame Perfect Equilibrium

The games described so far in this chapter have all had a very simple dynamic structure: they were either one-shot games or a repeated sequence of one-shot games. They also had a very simple information structure: each player in the game knew the other player's payoffs and available strategies, but did not know in advance the other player's actual choice of strategies. Another way to say this is that up until now we have restricted our attention to games with simultaneous moves.

But many games of interest do not have this structure. In many situations at least some of the choices are made sequentially, and one player may know the other player's choice before he has to make his own choice. The analysis of such games is of considerable interest to economists since many economic games have this structure: a monopolist gets to observe consumer demand behavior before it produces output, or a duopolist may observe his opponent's capital investment before making its own output decisions, etc. The analysis of such games requires some new concepts.

Consider for example, the simple game depicted in Table 5.6. It is easy to verify that there are two pure strategy Nash equilibria in this game, (Top, Left) and (Bottom, Right). Implicit in this description of this game is the idea that both players make their choices

Chapter 6

Theory of the Market

6.1 Introduction

In previous chapters, we studied the behavior of individual consumers and firms, describing optimal behavior when markets prices were fixed and beyond the agent's control. Here, we explore the consequences of that behavior when consumers and firms come together in markets. We will consider equilibrium price and quantity determination in a single market or group of closed related markets by the actions of the individual agents for different markets structures. This equilibrium analysis is called a **partial equilibrium** analysis because it focus on a single market or group of closed related markets, implicitly assuming that changes in the markets under consideration do not change prices of other goods and upset the equilibrium that holds in those markets. We will treat all markets simultaneously in the general equilibrium theory.

We will concentrate on modelling the market behavior of the firm. How do firms determine the price at which they will sell their output or the prices at which they are willing to purchase inputs? We will see that in certain situations the "price-taking behavior" might be a reasonable approximation to optimal behavior, but in other situations we will have to explore models of the price-setting process. We will first consider the ideal (benchmark) case of perfect competition. We then turn to the study of settings in which some agents have market power. These settings include markets structures of pure monopoly, monopolistic competition, oligopoly, and monopsony.

the firm to change its fixed factors so as to operate at a point of minimum average cost. But, if every firm tries to do this, the equilibrium price will certainly change.

In the model of entry or exit, the equilibrium number of firms is the largest number of firms that can break even so the price must be chosen to minimum price.

Example 6.3.2 $c(y) = y^2 + 1$. The break-even level of output can be found by setting

$$AC(y) = MC(y)$$

so that $y = 1$, and $p = MC(y) = 2$.

Suppose the demand is linear: $X(p) = a - bp$. Then, the equilibrium price will be the smallest p^* that satisfies the conditions

$$p^* = \frac{a}{b + J/2} \geqq 2.$$

As J increases, the equilibrium price must be closer and closer to 2.

6.4 Pure Monopoly

6.4.1 Profit Maximization Problem of Monopolist

At the opposite pole from pure competition we have the case of pure monopoly. Here instead of a large number of independent sellers of some uniform product, we have only one seller. A monopolistic firm must make two sorts of decisions: how much output it should produce, and at what price it should sell this output. Of course, it cannot make these decisions unilaterally. The amount of output that the firm is able to sell will depend on the price that it sets. We summarize this relationship between demand and price in a market demand function for output, $y(p)$. The market demand function tells how much output consumers will demand as a function of the price that the monopolist charges. It is often more convenient to consider the inverse demand function $p(y)$, which indicates the price that consumers are willing to pay for y amount of output. We have provided the conditions under which the inverse demand function exists in Chapter 2. The revenue that the firm receives will depend on the amount of output it chooses to supply. We write this revenue function as $R(y) = p(y)y$.

The cost function of the firm also depends on the amount of output produced. This relationship was extensively studied in Chapter 3. Here we have the factor prices as constant so that the conditional cost function can be written only as a function of the level of output of the firm.

The profit maximization problem of the firm can then be written as:

$$\max R(y) - c(y) = \max p(y)y - c(y)$$

The first-order conditions for profit maximization are that marginal revenue equals marginal cost, or

$$p(y^*) + p'(y^*)y^* = c'(y^*)$$

The intuition behind this condition is fairly clear. If the monopolist considers producing one extra unit of output he will increase his revenue by $p(y^*)$ dollars in the first instance. But this increased level of output will force the price down by $p'(y^*)$, and he will lose this much revenue on unit of output sold. The sum of these two effects gives the marginal revenue. If the marginal revenue exceeds the marginal cost of production the monopolist will expand output. The expansion stops when the marginal revenue and the marginal cost balance out.

The first-order conditions for profit maximization can he expressed in a slightly different manner through the use of the price elasticity of demand. The price elasticity of demand is given by:

$$\epsilon(y) = \frac{p}{y(p)}\frac{dy(p)}{dp}$$

Note that this is always a negative number since $dy(p)/dp$ is negative.

Simple algebra shows that the marginal revenue equals marginal cost condition can be written as:

$$p(y^*)\left[1 + \frac{y^*}{p(y^*)}\frac{dp(y^*)}{dy}\right] = p(y^*)\left[1 + \frac{1}{\epsilon(y^*)}\right] = c'(y^*)$$

that is, the price charged by a monopolist is a markup over marginal cost, with the level of the markup being given as a function of the price elasticity of demand.

There is also a nice graphical illustration of the profit maximization condition. Suppose for simplicity that we have a linear inverse demand curve: $p(y) = a - by$. Then the revenue function is $R(y) = ay - by^2$, and the marginal revenue function is just $R'(y) = a - 2by$. The marginal revenue curve has the same vertical intercept as the demand curve hut is twice as steep. We have illustrated these two curves in Figure 6.4, along with the average cost and marginal cost curves of the firm in question.

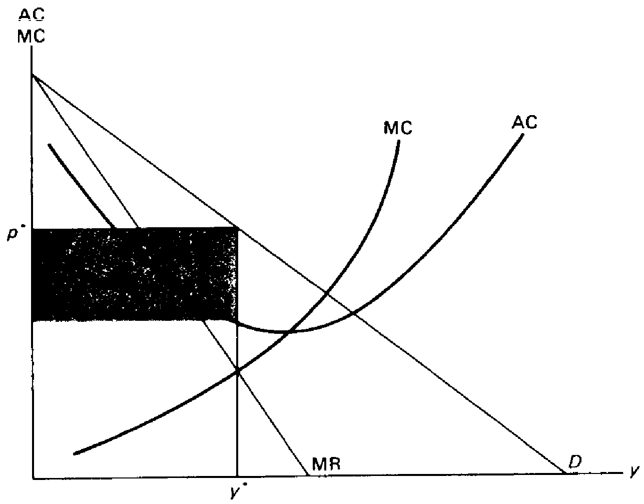

Figure 6.4: Determination of profit-maximizing monopolist's price and output.

The optimal level of output is located where the marginal revenue and the marginal cost curves intersect. This optimal level of output sells at a price $p(y^*)$ so the monopolist gets an optimal revenue of $p(y^*)y^*$. The cost of producing y^* is just y^* times the average cost of production at that level of output. The difference between these two areas gives us a measure of the monopolist's profits.

6.4.2 Inefficiency of Monopoly

We say that a situation is Pareto efficient if there is no way to make one agent better off and the others are not worse off. Pareto efficiency will be a major theme in the discussion of welfare economics, but we can give a nice illustration of the concept here.

Let us consider the typical monopolistic configuration illustrated in Figure 6.5. It turns out a monopolist always operates in a Pareto inefficient manner. This means that there is some way to make the monopolist is better off and his customers are not worse off.

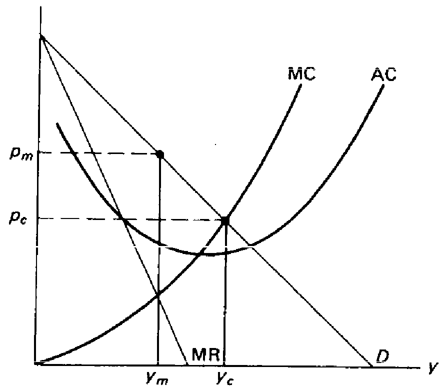

Figure 6.5: Monopoly results in Pareto inefficient outcome.

To see this let us think of the monopolist in Figure 6.5 after he has sold y_m of output at the price p_m, and received his monopolist profit. Suppose that the monopolist were to produce a small unit of output Δy more and offer to the public. How much would people be willing to pay for this extra unit? Clearly they would be willing to pay a price $p(y_m + \Delta y)$ dollars. How much would it cost to produce this extra output? Clearly, just the marginal cost $MC(y_m+\Delta y)$. Under this rearrangement the consumers are at least not worse off since they are freely purchasing the extra unit of output, and the monopolist is better off since he can sell some extra units at a price that exceeds the cost of its production. Here we are allowing the monopolist to discriminate in his pricing: he first sells y_m and then sells more output at some other price.

How long can this process be continued? Once the competitive level of output is reached, no further improvements are possible. The competitive level of price and output is Pareto efficient for this industry. We will investigate the concept of Pareto efficiency in general equilibrium theory.

6.4.3 Monopoly in the Long Run

We have seen how the long-run and the short-run behavior of a competitive industry may differ because of changes in technology and entry. There are similar effects in a monopolized industry. The technological effect is the simplest: the monopolist will choose the level of his fixed factors so as to maximize his long-run profits. Thus, he will operate where marginal revenue equals long-run marginal cost, and that is all that needs to be

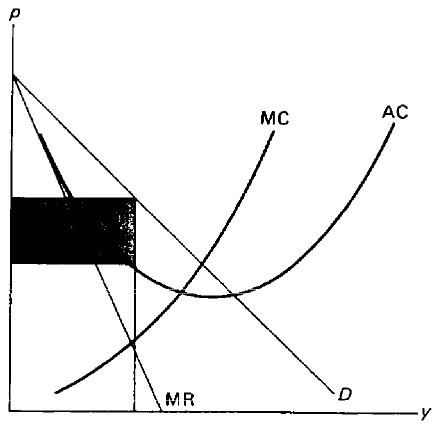

Figure 6.6: Short-run monopolistic competition equilibrium

or

$$p_i^* - \frac{c_i(y^*)}{y_i^*} \leqq 0 \quad \text{with equality if } y_i^* > 0 \quad i = 1, 2, \ldots, n.$$

Thus, the price must equal to average cost and on the demand curve facing the firm. As a result, as long as the demand curve facing each firm has some negative slope, each firm will produce at a point where average cost are greater than the minimum average costs. Thus, like a pure competitive firms, the profits made by each firm are zero and is very nearly the long run competitive equilibrium. On the other hand, like a pure monopolist, it still results in inefficient allocation as long as the demand curve facing the firm has a negative slope.

6.6 Oligopoly

Oligopoly is the study of market interactions with a small number of firms. Such an industry usually does not exhibit the characteristics of perfect competition, since individual firms' actions can in fact influence market price and the actions of other firms. The modern study of this subject is grounded almost entirely in the theory of games discussed in the last chapter. Many of the specifications of strategic market interactions have been clarified by the concepts of game theory. We now investigate oligopoly theory primarily from this perspective by introducing four models.

6.6.1 Cournot Oligopoly

A fundamental model for the analysis of oligopoly was the Cournot oligopoly model that was proposed by Cournot, an French economist, in 1838. A **Cournot equilibrium**, already mentioned in the last chapter, is a special set of production levels that have the property that no individual firm has an incentive to change its own production level if other firms do not change theirs.

To formalize this equilibrium concept, suppose there are J firms producing a single homogeneous product. If firm j produces output level q_j, the firm's cost is $c_j(q_j)$. There is a single market inverse demand function $p(\hat{q})$. The total supply is $\hat{q} = \sum_{j=1}^{J} q_j$. The profit to firm j is

$$p(\hat{q})q_j - c_j(q_j)$$

Definition 6.6.1 (Cournot Equilibrium) A set of output levels $q_1, q_2, \ldots, q_J$ constitutes a Cournot equilibrium if for each $j = 1, 2, \ldots, J$ the profit to firm j cannot be increased by changing q_j alone.

Accordingly, the Cournot model can be regarded as one shot game: the profit of firm j is its payoff, and the strategy space of firm j is the set of outputs, and thus a Cournot equilibrium is just a pure strategy Nash equilibrium. Then the first-order conditions for the interior optimum are:

$$p'(\hat{q})q_j + p(\hat{q}) - c_j'(q_j) = 0 \quad j = 1, 2, \ldots, J.$$

The first-order condition for firm determines firm j optimal choice of output as a function of its beliefs about the sum of the other firms' outputs, denoted by $\hat{q}_{-j}$, i.e., the FOC condition can be written as

$$p'(q_j + \hat{q}_{-j})q_j + p(\hat{q}) - c_j'(q_j) = 0 \quad j = 1, 2, \ldots, J.$$

The solution to the above equation, denoted by $Q_j(\hat{q}_{-j})$, is called the reaction function to the total outputs produced by the other firms.

Reaction functions give a direct characterization of a Cournot equilibrium. A set of output levels $q_1, q_2, \ldots, q_J$ constitutes a Cournot equilibrium if for each reaction function given $q_j = Q_j(\hat{q}_{-j})$ $j = 1, 2, \ldots, J$.

An important special case is that of duopoly, an industry with just two firms. In this case, the reaction function of each firm is a function of just the other firm's output. Thus, the two reaction functions have the form $Q_1(q_2)$ and $Q_2(q_1)$, which is shown in Figure 6.7. In the figure, if firm 1 selects a value q_1 on the horizontal axis, firm 2 will react by selecting the point on the vertical axis that corresponds to the function $Q_2(q_1)$. Similarly, if firm 2 selects a value q_2 on the vertical axis, firm 1 will be reacted by selecting the point on the horizontal axis that corresponds to the curve $Q_1(q_2)$. The equilibrium point corresponds to the point of intersection of the two reaction functions.

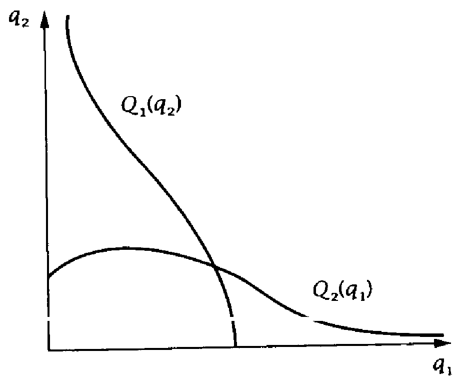

Figure 6.7: Reaction functions.

6.6.2 Stackelberg Model

There are alternative methods for characterizing the outcome of an oligopoly. One of the most popular of these is that of quantity leadership, also known as the Stackelberg model.

Consider the special case of a duopoly. In the Stackelberg formulation one firm, say firm 1, is considered to be the leader and the other, firm 2, is the follower. The leader may, for example, be the larger firm or may have better information. If there is a well-defined order for firms committing to an output decision, the leader commits first.

Given the committed production level q_1 of firm 1, firm 2, the follower, will select q_2 using the same reaction function as in the Cournot theory. That is, firm 2 finds q_2 to maximize

$$\pi_2 = p(q_1 + q_2)q_2 - c_2(q_2),$$

where $p(q_1 + q_2)$ is the industrywide inverse demand function. This yields the reaction function $Q_2(q_1)$.

Firm 1, the leader, accounts for the reaction of firm 2 when originally selecting q_1. In particular, firm 1 selects q_1 to maximize

$$\pi_1 = p(q_1 + Q_2(q_1))q_1 - c_1(q_1),$$

That is, firm 1 substitutes $Q_2(q_1)$ for q_2 in the profit expression.

Note that a Stackelberg equilibrium does not yield a system of equations that must be solved simultaneously. Once the reaction function of firm 2 is found, firm 1's problem can be solved directly. Usually, the leader will do better in a Stackelberg equilibrium than in a Cournot equilibrium.

6.6.3 Bertrand Model

Another model of oligopoly of some interest is the so-called Bertrand model. The Cournot model and Stackelberg model take the firms' strategy spaces as being quantities, but it seems equally natural to consider what happens if price is chosen as the relevant strategic variables. Almost 50 years after Cournot, another French economist, Joseph Bertrand (1883), offered a different view of firm under imperfect competition and is known as the Bertrand model of oligopoly. Bertrand argued that it is much more natural to think of firms competing in their choice of price, rather than quantity. This small difference completely change the character of market equilibrium. This model is striking, and it contrasts starkly with what occurs in the Cournot model: With just two firms in a market, we obtain a perfectly competitive outcome in the Bertrand model!

In a simple Bertrand duopoly, two firms produce a homogeneous product, each has identical marginal costs $c > 0$ and face a market demand curve of $D(p)$ which is continuous, strictly decreasing at all price such that $D(p) > 0$. The strategy of each player is to announce the price at which he would be willing to supply an arbitrary amount of the good in question. In this case the payoff function takes a radically different form. It is plausible to suppose that the consumers will only purchase from the firm with the lowest price, and that they will split evenly between the two firms if they charge the same price.

This leads to a payoff to firm 1 of the form:

$$\pi_1(p_1, p_2) = \begin{cases} (p_1 - c)x(p_1) & if \quad p_1 < p_2 \\ (p_1 - c)x(p_1)/2 & if \quad p_1 = p_2 \\ 0 & if \quad p_1 > p_2 \end{cases}.$$

Note that the Cournot game and the Bertrand game have a radically different structure. In the Cournot game, the payoff to each firm is a continuous function of its strategic choice; in the Bertrand game, the payoffs are discontinuous functions of the strategies. What is the Nash equilibrium? It may be somewhat surprising, but in the unique Nash equilibrium, both firms charge a price equal to marginal cost, and both earn zero profit. Formerly, we have

Proposition 6.6.1 *There is a unique Nash equilibrium (p_1, p_2) in the Bertrand duopoly. In this equilibrium, both firms set their price equal to the marginal cost: $p_1 = p_2 = c$ and earn zero profit.*

PROOF. First note that both firms setting their prices equal to c is indeed a Nash equilibrium. Neither firm can gain by rasing its price because it will then make no sales (thereby still earning zero); and by lowering its price below c a firm increase it sales but incurs losses. What remains is to show that there can be no other Nash equilibrium. Because each firm i chooses $p_i \geqq c$, it suffices to show that there are no equilibria in which $p_i > c$ for some i. So let (p_1, p_2) be an equilibrium.

If $p_1 > c$, then because p_2 maximizes firm 2's profits given firm 1's price choice, we must have $p_2 \in (c, p_1]$, because some such choice earns firm 2 strictly positive profits, whereas all other choices earns firm 2 zero profits. Moreover, $p_2 \neq p_1$ because if firm 2 can earn positive profits by choosing $p_2 = p_1$ and splitting the market, it can earn even higher profits by choosing p_2 just slightly below p_1 and supply the entire market at virtually the same price. Therefore, $p_1 > c$ implies that $p_2 > c$ and $p_2 < p_1$. But by stitching the roles of firms 1 and 2, an analogous argument establishes that $p_2 > c$ implies that $p_1 > c$ and $p_1 < p_2$. Consequently, if one firm's price is above marginal cost, both prices must be above marginal cost and each firm must be strictly undercutting the other, which is impossible.