# 1 Game Theory

## 1.1 Basic Elements of Noncooperative Games

**(i) The players defines who is involved**
**(ii) The rules: who moves when? What do they know when they move? What can they do?**
**(iii) The outcomes: given each player's actions, what are the related outcome?**
**(iv) The payoffs: Players' preferences (utility function) over the outcomes.**
**Example 7.B.1 Matching penny**

**Example 7.C.1: It is the same except that players move sequentially, say 1 puts her penny down first, then after seeing player 1's action, player 2 makes the decision.**

Insert graph here.

Example 7.C.3: **Matching penny version C: player 1 moves first and puts her penny down, but covered with her hands so that player 2 can not see her choice when making the decision.** How do we represent this game?

Insert graph here.

**Example: Standard matching penny. How can we represent this simplest game then? We actually can use the version C to represent it. What is the whole character of the standard one? Nobody knows the other's action when choosing their own actions. Look at the graph. Player 1 does not know what player 2 will do since 1 is moving first. Player 2 does not know what 1 will do because all of his decision nodes**

**are in the same information set. Of course, what is the other representation?another representation is to reverse 1 and 2.**

**Definition 7.C.1: A game is one of perfect information if each information set contains a single decision node. Otherwise, it is a game of imperfect information.**

### 1.2 Strategies and the Normal form representation of a Game

**Definition 7.D.1 Let $\mathcal{H}_i$ denote i's information sets, $\mathcal{A}$ the set of possible actions in the game, and $C(H) \subset \mathcal{A}$ the set of actions possible at information set $\mathcal{H}$. A strategy for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H), \forall H \in \mathcal{H}_i$**

$s_1$**: player H if 1 players H; player H if 1 players T.**
$s_2$**: player H if 1 players H; player T if 1 players T.**
$s_3$**: player T if 1 players H; player H if 1 players T.**
$s_4$**: player T if 1 players H; player T if 1 players T.**

**Definition 7.D.2: For a game with I players, the normal form representation $\Gamma_N$ specifies for each player i a set of strategies $S_i$ (with $s_i \in S_i$) and a payoff function $u_i(s)$ giving the vNM utility levels associated with the outcome arising from strategies profile $s$. Formally, we write $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$.**

### 1.3 Randomized choices

**Definition 7.E.1: Given player i's (finite) pure strategy set $S_i$, a mixed strategy for player i, $\sigma_i : S_i \to [0, 1]$, assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be player, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.**

**Definition 7.E.2: Given an extensive form game $\Gamma_E$, a behavior strategy for player i specifies, $\forall H \in \mathcal{H}$ and $a \in C(H)$, a probability $\lambda_i(a, H) \geq 0$, with $\sum_{a \in C(H) \lambda_i(a,H)=1}, \forall H \in \mathcal{H}_i$**

## 2 Simultaneous-Move Games

Insert graph here.

For this game, we can observe that it is always better to confess regardless what the other player does. As we know, there will be only one reasonable outcome: confess both. Thus confess is called a strictly dominant strategy. More generally,

**Defintion: $s_i \in S_i$ is a strictly dominant strategy for $i$ if $\forall s_i' \neq s_i$, we have $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}), \forall s_{-i} \in S_{-i}$.**

**Definition: $s_i \in S_i$ is strictly dominated for i if $\exists s_i' \in S_i$ such that $\forall s_{-i} \in S_{-i}, u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$. We also say $s_i'$ strictly dominates strategy $s_i$**

**Definition: $s_i \in S_i$ is weakly dominated if $\exists s_i' \in S_i$ such that $\forall s_{-i}$, $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ with strict inequality for some $s_{-i}$; it is a weakly dominant strategy if it weakly dominates every other strategy.**

**Iterated Deletion of Strictly Dominated Strategies**

Insert graph here.

This called the DA's brother game. The only different is that if both choose NC, then player can be freed as he has connection with the Judge. In this case, confess is no long a dominant strategy. And the elimination of strictly dominated strategy does not lead to a unique prediction.

However, we can do the following reasoning. Note that NC is strictly dominated for P2. Thus P2 will player confess for sure. However, given this, it is better for P1 to choose confess and we get CC as a unique prediction. One round elimination is more reasonable since it only requires that players are rational. However, for multiple round elimination, it requires much more information. For example, it requires that players know each other's payoffs. The more round there is, the more knowledge about the other player's rationality it requires. For example, for the second round elimination, it requires P1 to know P2 will rationally remove NC from the consideration.

**Definition: A mixed strategy $\sigma_i \in \Delta(S_i)$ is strictly dominated for player i in game $\Gamma_N = [I, \Delta(S_i), \{u_i(\cdot)\}]$ if there exists $\sigma_i'$ such that $\forall \sigma_{-i} \in \Pi_{j \neq i} \Delta(S_i)$,**

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

**In this case, we say that strategy $\sigma_i'$ strictly dominates $\sigma_i$. A strategy $\sigma_i$ is a strictly dominant strategy for player i if it strictly dominates all the other mixed strategies.**

## 3 Rationalizable Strategies:

**Definition: In a game with mixed strategy, $\sigma_i$ is a best response for i to his rivals' strategies $\sigma_{-i}$ if**

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \in \Delta(S_i)$$

**Strategy $\sigma_i$ is never a best response if there is no $\sigma_{-i}$ for which $\sigma_i$ is a best response.** $\sigma_i$ is a best response of $\sigma_{-i}$ if it is i's optimal choice when i beliefs that the others will will play $\sigma_{-i}$; it is never a best response if you can not find any belief on what other do that support it as an optimal choice.

**Definition: In a game with mixed strategy, $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player i's ratinalizable strategies.**

Insert graph here.

In the first round, we can first delete b4 as it is strictly dominated by playing b1 and b3 with equal probability. Then a4 can be eliminated because now it is strictly dominated by a2. Now, it can be verified that no more elimination can be done. The bad new is that the set of rationalizable strategies is always large and we need more assumptions to narrow down our prediction further and has to impose the requirement of equilibrium.

## 4 Nash Equilibrium

**Definition 8.D.1: $s = (s_1, \cdots, s_I)$ constitutes a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if $\forall i$,**

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i$$

Insert graph here.

Mm is Nash equilibrium. And it is the unique one. The rationalizable strategies contains all the strategies in the game. However, only Mm can be the Nash. Nash equilibrium is a subset of rationalizable strategies. The example illustrates that the concept of Nash equilibrium narrows down our prediction significantly in some games. But will NE always gives us a unique prediction? Unfortunately not. Consider the following example.

Insert graph here.

**Definition: i's best-response correspondence $b_i : S_{-i} \to S_i$ in the game with pure strategy is the correspondence that assigns to each $s_{-i} \in S_{-i}$ the set $b_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})\}, \forall s_i' \in S_i$**

**The strategy $s = (s_1, \cdots, s_I)$ is a NE of the game with pure strategy if and only f $s_i \in b_i(s_{-i}), \forall i = 1, \cdots, I$.**

**Mixed strategy Nash equilibria**

**Definition: A mixed strategy $\sigma = (\sigma_1, \cdots, \sigma_I)$ constitutes a NE of the game if $\forall i = 1, \cdots, I$,**

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \in \Delta(S_i).$$

**Example: Matching penny.**

**Proposition: Let $S_i^+ \subset S_i$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\sigma$. $\sigma$ is a NE in the game with mixed strategy iff $\forall i = 1, \cdots, I$,**

$$u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}), \forall s_i, s_i' \in S_i^+.$$

$$u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}), \forall s_i \in S_i^+, \ and \ s_i' \notin S_i^+.$$

**Corollary: A mixed strategy** $\sigma = (\sigma_1, \cdots, \sigma_I)$ **constitutes a NE of the game if** $\forall i = 1, \cdots, I$**,**

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}), \forall s_i' \in S_i.$$

**Why?**

**Corollary:** $s = (s_1, \cdots, s_2)$ **is a NE of** $\Gamma = [I, \{S_i\}, \{u_i(.)\}]$ **iff it is a mixed strategy NE of the game** $\Gamma = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$.

**Example: Mixed strategy equilibria in the Meeting in New York Game. (1/11,10/11) is an equilibrium.**

**Existence of NE**

**Proposition: Every game** $\Gamma = [I, \{\Delta(S_i)\}, \{u_i(.)\}]$ **in which every set** $\{S_i\}$ **is finite has a mixed strategy NE.**

**A NE exist in game** $\Gamma = [I, \{S_i\}, \{u_i(.)\}]$ **if** $\forall i$**,**
**(i)** $S_i$ **is a nonempty, convex, and compact subset of some Euclidean space** $\setminus^M$
**(ii)** $u_i(s)$ **is continuous in** $s$ **and quasi-concave in** $s_i$**.**

## 4.1 Games of Incomplete Information: Bayesian Nash Equilibrium

Here is an example:

**Example: Consider a modification of the DA's Brother game. With probability** $\mu$**, P2 has the preference type I, with the rest probability, P2 hates to rat his accomplice. In this case, he pays a psychic penalty equal to 6 years in prison for confessing. P1 always have the payoff in DA. In this game, the pure strategy of player 2 can be viewed as a function that for each possbile relization of this preference type indictes what action he will take. He has two information set and each information set has two actions. Thus, he has four strategies: (C if I, C if II), (C if I, NC if II), (NC if I, C if II), (NC if I, NC if II). For 1 has only one formational set, and two actions in the information set. Thus, he only has two strategies, C or NC.**

Insert graph here.

**Formally, in a Bayesian game, each player i's payoff function** $u_i(s_i, s_{-i}, \theta_i)$ **where** $\theta_i \in \Theta_i$ **is a random variable chosen by nature that is observed only by player** $i$**.** $F(\theta_1, \cdots, \theta_I)$ **is the commonly known joint probability distribution from which random variables are drawn from. Let** $\Theta = \Theta_1 \times \cdots \times \Theta_I$**, then we can summarize the Bayesian game as** $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$**.**

**Now we can define an ordinary (pure strategy) Nash equilibrium of this game of imperfect information, which is known as a Bayesian Nash equilibrium.**

**Definition: A (pure strategy) Bayesian Nash Equilibrium for the Bayesian game** $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ **is a profile of decision rules** $(s_1(\cdot), \cdots, s_I(\cdot))$ **that constitute a Nash equilibrium of game** $\Gamma_N = [I, \{\mathfrak{S}_i\}, \{\tilde{u}_i(\cdot)\}]$. That is $\forall i$,

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot)), \forall s_i'(\cdot) \in \mathfrak{S}_i.$$

**Proposition: A profile of decision rule** $(s_1(\cdot), \cdots, s_I(\cdot))$ **is a Bayesian Nash equilibrium in Bayesian game** $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ **iff,** $\forall i, \ and \ \bar{\theta}_i \in \Theta_i$ **occurring with positive probability**

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i(\theta_{-i})}, \bar{\theta}_i)|\bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s_i', s_{-i(\theta_{-i})}, \bar{\theta}_i)|\bar{\theta}_i], \forall s_i' \in S_i$$

It is better to consider an example.

**Example: First note that P2 with type I will play confess for sure as this is his dominant strategy. Likewise, P2 with type II will play NC which is a dominate strategy. Given P2's strategy, P1's best response is to play NC if** $-10\mu + 0(1-\mu) > [-5\mu - 1(1-\mu)]$ **or equivalently if** $\mu < 1/6$ **and player confess if** $\mu > 1/6$**. He is in different if equal.**

**Example: Consider two persons are in a team. Their is a task. If it is finished then I will award a prize to you. You may value the prize differently**
**How to solve the Baysian Nash? We know each of their action depends on their type** $s_i(\theta_i)$**. It takes value 1 or 0. For player 1, given player 2 acts according to** $s_2(\theta_2)$**, his payoff to develope is** ${\theta_i}^2 - c$**, and 0 is** $\theta_1^2 prob(s_2(\theta_2) = 1)$**. Therefore, firm 1's best response is to develop iff** $\theta_1 \geq (\frac{c}{1-prob(s_2(\theta_2)=1)})^{0.5}$**. This means the best response of each firm is a cutoff rule. Suppose** $\hat{(\theta)}_1, \hat{(\theta)}_2$ **are players' rules. Then from the above equation, we**

**have:**

$$\hat{\theta}_1^2\hat{\theta}_2 = c$$

**. Similarly, for player 2, we have**

$$\hat{\theta}_2^2\hat{\theta}_1 = c$$

**. Now we can solve $\hat{\theta}_1 = \hat{\theta}_2 = c^{1/3}$**

### 4.2 Trembling-hand Perfection

**Given the normal form game $\Gamma_N[I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, we can define a perturbed game $\Gamma_\epsilon[I, \{\Delta_\epsilon(S_i)\}, \{u_i(\cdot)\}]$ by choosing for each player i and strategy $s_i \in S_i$ a number $\epsilon(s_i) \in (0, 1)$ with $\sum_{s_i \in S_i} \epsilon_i(s_i) < 1$ and define**

$$\Delta_\epsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \epsilon(s_i), \forall s_i \in S_i, and \sum_{s_i \in S_i} \sigma_i(s_i)\} = 1$$

**Definition: A NE in $\Gamma_N$ is a trembling hand perfect if $\exists$ some sequence of perturbed games $\{\Gamma_{\epsilon^k}\}_{k=1}^{\infty}$ that converges to $\Gamma_N$, for which $\exists$ some associated sequence of Nash equilibrium $\{\sigma^k\}_{k=1}^{\infty}$ that converges to $\sigma$.**

**Proposition: A NE $\sigma$ of game $\Gamma_N[I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is trembling hand perfect iff $\exists$ some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \sigma^k = \sigma$ and $\sigma_i$ is a best response to every element of the sequence $\{\sigma_{-i}^k\}_{k=1}^{\infty}$.**

**If $\sigma$ is a trembling hand perfect NE, then $\sigma_i$ is not a weakly dominated strategy for any player. Hence in any trembling hand perfect NE, no weakly dominated pure strategy can be played with positive probability.**

## 5 An introduction to Auctions

### 5.1 Private value auctions

**(i)private values: everyone's valuation only depends on their own signals**
**(ii) independently and identically distributed (IID)**

### 5.2 The symmetric model

**Basic elements of the model:**
**(i) One risk neutral seller with single object for sale**
**(ii)N potential risk neutral bidders**
**(iii) Each buyer assigns a value $X_i$ to the object**

**(iv) $X_i$ is independently and identically distributed on $[0, w]$ with $c.d.f F(\cdot)$ and $p.d.f f(\cdot)$. (v) Information structure: i knows his realization of $x_i$ but does not observe others' valuations and only know they are independently distributed according to F. (vi) The structure of the game is common knowledge.**

## 5.3 Second-price auctions

A second price auctions means the highest bidder gets the object and pays the second highest bid.

**If bidder i bids $b_i$, the payoffs are**

$$\Pi_i = \begin{cases} x_i - \max_{j \neq i} b_j & if\ b_i \geq \max_{j \neq i} b_j \\ 0 & if\ b_i < \max_{j \neq i} b_j \end{cases}$$

If more than one bids the same highest bid, then each wins with the same probability. Given the game structure, what is the equilibrium of this game?

**Proposition: In a second-price sealed-bid auctions, it is a weakly dominant strategy to bid according to $\beta^{II}(x) = x$.**

**Proof: consider player 1. Suppose the highest bid among others is $p_1$. If $p_1 \geq x_1$, then bidding $x_1$ yields zero; bidding less than $x_1$ yields 0; bidding higher than $p_1$ yields negative payoff. If $p_1 < x_1$, then bidding $x_1$ yields positive payoff; bidding greater than $p_1$ yields the same payoff as bidding $x_1$; bidding less than $p_1$ will yield zero payoff if it is less than $p_1$ and yields the same payoff if it is greater than $p_1$. Therefore, bidding $x_1$ is a weakly dominant strategy.**

This means everybody adopts the above strategy is a Bayesian Nash. But of course, are there any other equilibrium? Consider two bidders, one bidder always bid zero and the other always bid w is an equilibrium. But of course, this equilibrium is not symmetric and not robust to trembling hand refinement.

## 5.4 First price auctions

**In first price auctions, each bidder submits a bid and the one with the highest bid wins and pays that bid.**

$$\Pi_i = \begin{cases} x_i - b_i & if\ b_i \geq \max_{j \neq i} b_j \\ 0 & if\ b_i < \max_{j \neq i} b_j \end{cases}$$

Again, ties are broken evenly. What strategies should we expect bidders to play.

How should we solve the equilibrium of this game? If you think about the game, it is quite complicated since the bidding strategy can be anything and is not coutable.

**In pratice, we restrict our attentions to certain structure of the equilibrium. Since buyers are symmetric, we can focus on the symmetric equilibrium in the sense that everyone will adopt the same bidding strategy, say $\beta(x)$. Within this class, we look for stirctly increasing function. We need to pin the bidding function. Consider player 1. If all the other players follows $\beta(x)$, what is his best response. First note that he will not bid more than $\beta(w)$ since bidding $\beta(w)$ is better. Second, a bidder with value zero will never submit a positive bid since we would make a loss if he were to win. Thus, $\beta(0) = 0$.**

**Therefore, if player 1 bids $b \leq \beta(w)$, his payoff is**

$$(x - b)F(\beta^{-1}(b))^{N-1}$$

**Why, he wins the auction only when he submits the highest bid. The FOC is then,**

$$-F(\beta^{-1}(b))^{N-1} + (x - b)(N - 1)(\beta^{-1}(b))^{N-2} f(\beta^{-1}(b)) \frac{1}{\beta'(\beta^{-1}(b))} = 0$$

**In a symmetric equilibrium, player 1 will indeed follow the bidding function, thus $b = \beta(x)$. Now, we have**

$$-F(x)^{N-1} + (x - \beta(x))(N - 1)(x)^{N-2} f(x) \frac{1}{\beta'(x)} = 0$$

$$\begin{aligned}
\beta'(x)F(x)^{N-1} + \beta(x))(N - 1)(x)^{N-2} f(x) &= x(N - 1)(x)^{N-2} f(x) \\
(\beta(x)F(x)^{N-1})' &= x(N - 1)(x)^{N-2} f(x) \\
\beta(x) &= \frac{1}{F(x)^{N-1}} \int_0^x ydF(y)^{N-1}
\end{aligned}$$

**Denote $z = \beta^{-1}(b)$, then given the bidding function, player 1's payoff is**

$$\begin{aligned}
\Pi(b, x) = \Pi(\beta(z), x) &= (x - \frac{1}{F(z)^{N-1}} \int_0^z ydF(y)^{N-1})F(z)^{N-1} \\
&= xF(z)^{N-1} - \int_0^z ydF(y)^{N-1} \\
&= xF(z)^{N-1} - zF(z)^{N-1} + \int_0^z yF(y)^{N-1} dy \\
&= (x - z)F(z)^{N-1} + \int_0^z F(y)^{N-1} dy
\end{aligned}$$

$$\Pi(\beta(x), x) - \Pi(\beta(z), x) = F(z)^{N-1}(z - x) - \int_x^z F(y)^{N-1} dy$$
$$= \int_x^z (F(z)^{N-1} - F(y)^{N-1}) dy \geq 0$$

**For uniform distribution on [0,1], the bidding function is** $\beta^I(x) = \frac{N-1}{N}x$

# 6 Dynamic Games

## 6.1 Sequential Rationality, Backward Induction, and Subgame Perfection

As in the previous introduction, there are Nash equilibrium that is not reasonable in dynamic setting. This brings us the concept of subgame perfect Nash equilibrium.

**Example: Here is a game similar to what we have introduced. There are two firms in the market. One entrant and one incumbent. The game is played as follows. The entrant first decide whether to enter or stay out. Then the incumbent can choose whether to fight by cutting the price or accommodate by maintaining the price and gives up some market share to the entrant.**

Insert graph here.

**There are two Nash. However the (out, fight if in) is not a sensible prediction for Entrant. This is because if the entrant really enters, the incumbent should not fight. To rule out such non-sensible prediction, we want to assume that players must satisfy the principle of sequential rationality. That is he should act optimally whenever he is asked to move. Actually, how could we select the sensible Nash. We can do this, we can first determine the optimal behavior of the incumbent if he is chosen to move. Then given his optimal choice, we determine the entrant's choice given that he can anticipate the incumbent's optimal decision later on. Such a procedure is called backward induction in the sense that solving the optimization problem from the very end and going backward.**

## 6.2 Backward Induction in Finite games of perfect information

**Example: Consider the following game:**

Insert graph here.

**Proposition: (Zermelo's Theorem) Every finite game of perfect information has a pure strategy NE derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then such equilibrium is a unique.**

## 6.3 SPNE

Insert graph here.

**This is game similar to the predation game but now both firms need to choose whether to fight or accommodate simultaneously. There are three pure strategy NE: ((O, A if in), (F if I)), ((O, F if I),(F if I)), ((I,A if I),(A if I)). However, given that the entrant is in, in the simultaneous moving game, there is a unique Nash, i.e. (A,A). But given the prediction, the entrant should choose to enter. Such a reasoning is captured by the notion of SPNE. Before introducing the concept, we need to know what is called a subgame.**

**Definition: A subgame of $\Gamma_E$ is a subset of the game having the following properties: (i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors of the node (both immediate and later), and contains**

**only these nodes.**
**(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x.**

**Definition:** $\sigma = (\sigma_1, \cdots, \sigma_I)$ **is a SPNE in** $\Gamma_E$ **if it induces a NE in every subgame of** $\Gamma_E$.

**How to find the SPNE.**
**(i) Start at the end of the game tree, and identify the NE for each of the final subgames.**
**(ii) Select on Nash in each of these final subgames, and derive the reduced extensive form game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.**
**(iii) Repeat the last two steps for the reduced game. Continue the procedure until every move in** $\Gamma_E$ **is determined.**
**(iv) If multiple equilibria are never encountered in any step of this process, this profile of strategies is the unique SPNE. If multiple equilibria are encounted, the full set of SPNEs is identified by repeating the procedure for each possible equilibrium that could occur for the subgames in question.**

Insert graph here.

**This is modified version of the predation game. There are two different markets. One big and one small. If they both choose the same market, then both suffer. If different, then the one in the large earns positive but small with negative. There are two NE in the subgame: ((in,large if in),(small if in)) and ((out, small if in),(large if in))**

## 6.4 Beliefs and Sequential Rationality

Although SPNE is a useful way to capture the principle of sequential rationality, it is usually not enough. Consider the following game:

**Example: there are two strategies the entrant can enter and the incumbent can not tell which one. There are two pure strategy NE here: (out, F if in) and (in1, A if in)). For this game, both NE are SPNE since the whole game is the only subgame.**

**However, the first one does not seem very reasonable. When Firm 1 moves, he is always better to accommodate when entry occurs. How could we rule it out? We will use the notion of weak perfect Bayesian equilibrium. It requires that at any point in the game, a player's strategy prescribe optimal actions from that point on given her opponents' strategies and her beliefs about what has happened so far in the game and that her beliefs be consistence with the strategies being played. To present this idea formally, we need to introduce two concepts first: system of beliefs and sequential rationality of strategies.**

Insert graph here.

**Definition: A system of belief** $\mu$ **in** $\Gamma_E$ **is a specification of a probability** $\mu(x) \in [0,1]$ **for each decision node x such that** $\sum_{x \in H} \mu(x) = 1, \forall H$**.**

**A strategy** $\sigma$ **in** $\Gamma_E$ **is sequentially rational at** $H$ **given a system of beliefs** $\mu$ **if, denoting by i(H) the player who moves at** $H$**, we have**

$$E[u_{i(H)}|H, \mu, \sigma_{i(H)}, \sigma_{-i(H)}] \geq E[u_{i(H)}|H, \mu, \tilde{\sigma}_{i(H)}, \sigma_{-i(H)}], \forall \tilde{\sigma}_{i(H)} \in \Delta(S_{i(H)})$$

**If strategy profile** $\sigma$ **satisfies this condition for all possible information sets** $H$**, then we say that** $\sigma$ **is sequentially rational given** $\mu$**.**

**Definition: A profile of strategies and system of beliefs** $(\sigma, \mu)$ **is a weak perfect Bayesian equilibrium (weak PBE) in** $\Gamma_E$ **if it has the following properties:**
**(i)** $\sigma$ **is sequentially rational given belief system** $\mu$ **(ii)** $\mu$ **is derived from** $\sigma$ **through Bayes' rule whenever possible. That is, for any information set H such that** $prob(H|\sigma)$**, we must have** $\mu(x) = \frac{Prob(x|\sigma)}{Prob(H|\sigma)}, \forall x \in H$**.**

**Proposition:** $\sigma$ **is a NE of extensive form game** $\Gamma_E$ **iff** $\exists \mu$ **such that,**
**(i)** $\sigma$ **is sequentially rational given** $\mu$ **at all** $H$ **with** $Prob(H|\sigma) > 0$
**(ii)** $\mu$ **is derived from** $\sigma$ **through Bayes' rule whenever possible.**

**Example: accommodate if entry occurs must be player in any weak PNE since this is the optimal choice for any belief. Thus, the NE (out, fight if entry occurs) is ruled out.**

**Example: Consider the following 'joint venture' entry game. Now there is another**

**entrant 2. Here is the story. FE1 can choose to enter with two options. He can choose to enter by his own or propose a joint venture with the other entrant FE2. If it propses a joint venture, FE2 can either accept or decline. If E2 accepts, then E1 enters with E2's assistance. If not, then E1 must decide whether to enter on its own. The incumbent can observe whethere E1 has entered but not whether it is with E2's assistance. Fighting is the best response for I if E1 is unassisted, but not if assisted. Finally, if E1 is unassisted, it wants to enter only if the incumbent accommodates; but if E1 is assisted by E2, then because it will be such a strong competitor, its entry is profitable regardless of whether the incumbent fights.** Insert graph here.

To identify the weak PBE, first consider this information set. E2 must accept the joint venture if firm E1 proposes it. But if so, then in any weak PBE, firm E1 must propose the joint veture. Next, these two conclusion imply that firm I 's information set is reached with positive porbability in any weak PBE. Applying Bayesian updating, at this infomration set, we conclude that the beliefs at this information set must assign a probability of 1 to being at the middle node. . THus, firm I must be accomodate if entery occurs. Finally, E1 must enter.

(propose joint venture, in if E2 declines), (accept), (A if In) and belief system $\mu$ with middle nodes for sure is a weak PBE. Note that this is not the only NE though. For example: (out, out if E2 declines), (decline), (fight if entery occurs).

**Strengthening of the weak PNE**

The weak PNE requires that consistency only for the belief on the equilibrium path. There is no restriction on the off equilibrium path. In some situation putting on some restrictions on the off-equilibrium path beliefs can narrow down the equilibrium even further. Here is an example:

Insert graph here.

**In this game, Nature firm draw a status with half half probability. The information set over here means that neither player can observe the nature. Then player 1 choose either x or y. Only when y is chosen, player 2 moves by choosing l and r. Consider the dipictured strategies. This is a weak PBE given the belief. However, this belief does not seem quite reasonable. Player 2's information set can be reached only when 1 player y. But this deviation should be independent of the nature. Thus, it seems it should be more reasonable to assign half half to the belief. But given this, l is no longer optimal for 2.**

**Consider the following example:**

Insert graph here.

**The strategy depictured is a weak PBE but a SPNE. Again, since this information set is not achieved with positve probability, we can assign any belief. However, the strategy does not consitute a Nash in the subgame. These two examples illustrate that maybe the concept of weak PNE is too weak. Now we will introduce a more restrictive equilibrium concept developed by Kreps and Wilson two decades ago.**

**Definition:** $(\sigma, \mu)$ **is a sequential equilibrium (SE) of** $\Gamma_E$ **if**
**(i)**$\sigma$ **is sequentially rational given** $\mu$
**(ii)**$\exists$ **a sequence of fully mixed strategies with** $\{\sigma^k\}_{k=1}^{\infty} \to \sigma$**, such that** $\mu^k \to \mu$**, where** $\mu^k$ **denotes the beliefs derived from strategy profile** $\sigma^k$ **using Bayes' rule.**

Some remarks. First SE must be weak PBE. Why? For weak PBE, it only requires the consistency for the on-path information set, i.e., Bayes' rule. But the limit for (ii) is just this requirement. However, the reverse is not right.

**Consider the first example: For any totally mixed strategy, the only belief is half half in player 2's information set. Given this, 2 must play r and player 1 must player y. This is the unique SE.**

**Consider the second example: Consider any totally mixed strategy and any node in firm 1's information set. Suppose this is** $p1$ **and** $p2$ **here, then the belief of accommodate is exactly** $p2$**. Therefore, it must consist a NE in the subgame.**

**Proposition:** $SE \subseteq SPNE, weakPBE \subseteq NE$

## 6.5 Reasonable beliefs and forward induction

We saw some importance of beliefs to narrow down the prediction of outcome. Now we talk more about it. Consider the following example:

Insert graph here.

**The first one is a variant of the entry game where firm I would now find it worthwhile to fight if it knew that the entrant chose strategy in1; the second is a variant of the Niche Choice game where firm E now targets a niche at the time of its entry. Depictured are weak PBE. They are also SE. Why? First, given the belief system, the strategies are sequential rational. Second the following fully mixed strategy can justify the belief:** $1-\epsilon-\epsilon^2, \epsilon, \epsilon^2$**. This belief is not quite reasonable. If enter occurs, firm E should propose IN2 as this is better regardless what Firm I does. Consider the following speed, if I am in I certainly will choose IN2 as it is always better for me. Think about your strategy carefully. For the next example, Note that SN is strictly dominate by out. The firm can say the only way I could do better by entering is choosing LN. Both of the two examples is something like strictly dominance.**

Consider the next example: Insert graph here.

**We know there are two SPNE in this game. First, both are weak PBE and SE. The forward induction can actually rule of the EQ with (small if in). The strategy**

**in and SN is strictly dominated by staying out. If in happens then, E must choose large market. Thus, the incumbent should choose SN. But such reasoning may be problemsome if players make mistake. If in already happened, the I will say forget about it and I think you just made a mistake and even not I will target the LN.**

# 7 Bargaining games

**Bargaining game is a game with perfect information. Consider the very simple example. There are two players, 1 and 2 bargaining to determine the split of v dollars. Here is the rule: the game begins in period 1; in period 1, player 1 makes an offer of a split to player 2, which player 2 may then accept or reject. If accepts, the propsed split is immediatetly implemented and the game ends. If she rejects, nothing happens until period 2. In period 2, the players' roles are revered. Each player has a discount factor $\sigma$. However, after some finite number of periods T, if an agreement has not yet been reached, then both get zero. This is a very nice game of perfect information. We would like to see how the concept of SPNE can predict the outcome. Assume that player will accept if he is indifferent.**

**First consider T=1. Then the SPNE is (v,0). Now consider two periods. First look at the second period. Player will propose (v,0). Player 2's payoff is $\sigma v$. Back to period 1, player 1 is deciding what offer, if he gives player 2 less than $\sigma v$, player 2 will reject and leave zero to player 1 once reached. Thus, he will just propose $\sigma v$ to induce player 2 to accept.**

Insert graph here.

**It looks to us that who will make the last proposal is crucial in the game. First consider when $T$ is odd. Then player 1 makes the offer in the last period if no previous agreement has been reached. The payoff is then $(\sigma^{T-1}v, 0)$. Now consider play in the subgame starting in T-1 when no previous agreement has been reached. Player 2 makes the offer in this period. In any SPNE, player 1 will accept an offer in T-1 iff it provides her with payoff $\sigma^{T-1}v$ since otherwise she will do better rejecting it and waiting to make an offer in period T. Given this fact, in any SPNE, player 2 must make an offer $\sigma^{T-1}v$ to player 1 who will then accept it. The payoffs arising if the**

game reaches T-1 must therefore be $(\sigma^{T-1}v, \sigma^{T-2}v - \sigma^{T-1}v)$.

Continuing in this fashion, we can determine that the unique SPNE when T is odd results in an agreement being reached in period 1, a payoff for player 1 of

$$v_1(T) = v(1 - \sigma + \sigma^2 - \cdots + \sigma^{T-1}) = v[(1-\sigma)\frac{1-\sigma^{T-1}v}{1-\sigma^2} + \sigma^{T-1}]$$

and a payoff to player 2 of $v_2^*(T) = v - v_1^*(T)$.

If T is even, then player 1 must earn $v - \sigma v_1^*(T-1)$ because in any SPNE, player 2 will accept an offer in period 1 iff it gives her at least $\sigma v_1^*(T-1)$, and player 1 will offer her exactly this amount.

Infinite horizon bilateral bargaining

To start, let $\bar{v}_1$ denote the largest payoff that player 1 gets in any SPNE. Given the stationarity of the model, this is also the largest amount that player 2 can expect in the subgame that begins in period 2 after her rejection of player 1's period 1 offer. As a result, player 1's payoff in any SPNE cannot be lower than the amount $\underline{v}_1 = v - \sigma\bar{v}_1$.

Next, we claim that, in any SPNE, $\bar{v}_1$ cannot be larger than $v - \sigma\underline{v}_1$. Note that in any SPNE, player 2 is certain to reject any offer in period that gives her less than $\sigma\underline{v}_1$ because she can earn at lest this amount by rejecting it and waiting to make an offer in period 2. Thus, player 1 can do no better than $v - \sigma\underline{v}_1$ by making an offer that is accepted in period 1. Thus, we must have $\bar{v}_1 \leq v - \sigma\underline{v}_1$.

The above two things imply that $\underline{v}_1 = \bar{v}_1$, and player 1's SPNE payoff is uniquely determined. Denote this payoff by $v_1^0$. Since $v_1^0 = v - \sigma v_1^0$, we have $v_1^0 = \frac{v}{1+\sigma}$ and player 2 must earn the rest $\frac{\sigma v}{1+\sigma}$. Note that this outcome is the same as that in finite version and when T goes to infinity.

# 8 Market power

## 8.1 Monopoly pricing

Profit maximizing monopolist with single goods.
Demand function is $x(p)$
Production cost: $c(q)$

The monopoly's problem:

$$Max_p \quad px(p) - c(x(p))$$

**We can also write the problem as if the seller is choosing quantity.**

$$Max_{q \geq 0} p(q)q - c(q)$$

**The FOC then gives us:**

$$p'(q^m)q^m + p(q^m) \leq c'(q^m), with\ equality\ if\ q^m > 0$$

**We usually propose some conditions to guarantee that the FOC characterize the optimality, i.e.,**

$$p'(q^m)q^m + p(q^m) = c'(q^m)$$

**This implies $p(q^m) > c'(q^m)$, and so the price under monopoly exceeds marginal cost.**

## 8.2 Static models of Oligopoly

**Two firms: 1 and 2**
**demand is given by $x(p)$**
**Constant marginal cost $c > 0$**
**Game structure: simultaneous moving game by naming a price**
**Sales for firm j are then given by**

$$x_j(p_j, p_k) = \begin{cases} x(p_j) & if\ p_j < p_k \\ 1/2x(p_j) & if\ p_j = p_k \\ = 0 & if\ p_j > p_k \end{cases} \tag{1}$$

What is the equilibrium of the model.

**Proposition: $p_1^* = p_2^* = c$ is the unique NE.**

**It is straightforward to verify that the proposed strategies are NE. Given seller 1 is pricing at c, for seller 2, setting c gives him zero; setting higher than c gives him zero; setting lower than c gives him negative payoff.**
**Suppose $p_1 < c$, then the best response for seller 2 is setting $p_2 = p_1 + \epsilon$. However, given this, it is no longer optimal for firm 1 to set $p_1$. Suppose $p_1 > c$, then it is optimal for seller 2 to set any amount $p_2 < p_1$. However, given any $p_2$, it is not optimal for seller 1 to set price at $p_1$.**

## 8.3 Capacity Constraint and Decreasing returns to scale

**Let $\bar{q} = 3/4x(c)$. In this case, $p_1^* = p_2^* = c$ is no longer an equilibrium. To see this, note that because firm 2 cannot supply all demand at price c, firm 1 can anticipate this and increase the price by a little bit. In this case, his profit is positive which provides him with the incentive to deviate. In fact, as long as the capacity is lower the the whole**

**demand, the result breaks down. What is the equilibrium in such a game? You are asked to answer it in the assignment.**

## 8.4 Product differentiation

Suppose there are $J > 1$ firms. Each firm produces at a constant marginal cost $c > 0$. The demand for firm j's product is given by the continuous function $x_j(p_j, p_{-j})$. In a simultaneous pricing game, each firm maximize his profit by taking rivals' price as $\bar{p}_{-j}$, i.e.,

$$Max_{p_j}(p_j - c)x_j(p_j, \bar{p}_{-k})$$

The FOC gives $x_j + (p_j - c)\frac{\partial x_j}{\partial p_j} = 0$. Thus, as long as $x_j > 0$, firm j's best response must have $p_j > c$. The implication is that in the presence of product differentiation, equilibrium prices will be above the competitive level. The question now is how could we provide a micro function for such a demand function. Here is the famous Hotelling model, also called the linear city model.

**Example: The linear city model of product differentiation. Consider a city that can be represented as lying on a line segment of length 1. There is a continuum of consumers whose total number is M and uniformly distributed along the line. Let z be the consumer's location index and denote the distance from the left end of the city. At each end of the city is located one supplier: firm 1 is at the left end and firm 2 at the right. The two firms are producing the same product at constant marginal cost $c > 0$. Every consumer need only one unit of the product. The total cost of buying from firm j for a consumer located a distance d from firm j is $p_j + td$.**

Insert graph here.

**Given $p_1$ and $p_2$, let $\hat{z}$ be defined as the indifferent type:** $p_1 + t\hat{z} = p_2 + t(1 - \hat{z}) \Leftrightarrow \hat{z} = \frac{t+p_2-p_1}{2t}$**.**
**Thus, we have**

$$x_1(p_1, p_2) = \begin{cases} 0 & if\ p_1 > p_2 + t \\ (t + p_2 - p_1)/2t & if\ p_1 \in [p_2 - t, p_2 + t] \\ M & if p_1 < p_2 - t \end{cases} \tag{2}$$

**By the symmetry of the two firms, the demand function for firm 2 is**

$$x_2(p_1, p_2) = \begin{cases} 0 & if\ p_2 > p_1 + t \\ (t + p_1 - p_2)/2t & if\ p_2 \in [p_1 - t, p_1 + t] \\ M & if p_2 < p_1 - t \end{cases} \tag{3}$$

**Now for firm 1, his optimization problem is then**

$$\max_{p_1} (p_1 - c)x_1(p_1, p_2)M$$

**However, the best response must lies in the interval. For example, for firm 1, charging a price more than $p_2 + t$ is the same as charging $p_2 + t$ which yields zero payoff; charging a price less than $p_2 - t$ is dominated by charging $p_2 - t$. Thus, Firm 1's problem is to maximize**

$$\max_{p_1} (p_1 - c)(t + p_2 - p_1)M/2t$$

**The FOC gives us**

$$b(p_2) = \begin{cases} p_2 + t & if\ p_2 \leq c - t \\ (t + p_2 + C)/2t & if\ p_2 \in (c - t, c + 3t) \\ p_2 - t & if p_2 \geq c + 3t \end{cases} \tag{4}$$

**Insert graph here.**

**Therefore, the unique equilibrium is the symmetric one with $p_i = p_j = c + t > c$.**

# 9 Repeated Interaction

**$\sigma$ is the discounting factor.** Consider the following strategy for firms (**Nash reversion strategy**):

$$p_{jt}(H_{t_1}) = \begin{cases} p^m & if\ all\ element\ in\ H_{t-1}\ equal\ (p^m, p^m)\ or\ t = 1 \\ c\ otherwise & \end{cases} \tag{5}$$

In words, firm j's strategy is to play the monopoly price $p^m$ in the first period, and play $p^m$ in in every pervious period both firms have charged price $p^m$ and otherwise charges a price equal to zero. This strategy is called the Nash reversion strategy. Firms cooperate until someone deviates. That means if anyone deviate, the outcome will be the Bertand forever thereafter. It turns out we have the following proposition.

**Proposition: The Nash reversion strategy constitute a SPNE of the infinitely repeated Bertrand duopoly game iff $\sigma \geq 0.5$.**
**Proof: Recall that SPNE requires that the strategy consistute a NE in every subgame. We know for each subgame of the whole game it is a repeated Bertrand duopoly game exactly like the game as a whole. Therefore, we only need to show that for any previous history of play, the strategy is a NE of an infinitely repeated Bertrand game. However, given the structure of the strategy, we only need to care about two types of previous histories: one is the one with previous deviation and the other is the one without previous deviation.**
**Consider first a subgame with previous deviation. Then strategy says they will set price at marginal cost forever regardless of the rival's behavior. This is of course a NE because each firm can earn at most zero when his opponent always sets price at c, which is same as following the prediction.**
**Now consider a subgame without no previous deviations. Each firm j knows that its rival's strategy will charge $p^m$ until he deviates and will charge c forever thereafter.**
**If j follows the equilibrium he earns $1/2(p^m - c)x(p^m)\frac{1}{1-\sigma}$. If he deviates in the first period of the subgame he will set price a little bit small that $p^m$ and capture the whole market in this period and at c thereafter. The payoff is therefore arbitrarily close to $(p^m - c)x(p^m)$. He will not deviate in this way iff**

$$1/2(p^m - c)x(p^m)\frac{1}{1-\sigma} \geq (p^m - c)x(p^m) \Leftrightarrow \sigma \geq 0.5$$

**. The analyze for the strategy if decides to deviate starting from period t in the subgame is exactly the same argument. This completes the proof.**

Although the Nash reversion strategy constitute an SPNE when $\sigma \geq 0.5$, they are not the only SPNE. This is summarized in the following lemma.

**Proposition: In the infinitely repeated Bertrand duopoly game, when $\sigma \geq 0.5$ repeated choice of any price $p \in [c, p^m]$ can be supported as a SPNE outcome. By contrast, when $\sigma < 0.5$, any SPNE must have all sales occurring at a price equal to c in every period.**

**In an infinitely repeated game, any feasible discounted payoffs that give each player, on a per-period basis, more than the lowest payoff that he could get in a single play of simultaneous move game can be sustained as the payoffs of an SPNE if players discount the future to a sufficiently small degree.**

## 9.1 Entry

**Stage 1: All potential firms simultaneously decide in or out. There is a fixed cost K¿ for entry.**
**Stage 2: All firms entered play some oligopoly game.**

The oligopoly game could be any game such as Cournot, Bertrand or modified Bertrand. Note that the second stage is a subgame exactly the same as we have studied before with fixed number of firms. We shall assume that in this subgame, there is a unique symmetric equilibrium in the second stage. This is reasonable since we are looking at identical firms. For example, for Cournot competition, both firm will produce $\frac{a-c}{3}$. With Bertrand, both firm will set price at marginal cost. With linear city, both firm will price at $c+t$. Therefore we can denote the profit of **a firm in the second stage equilibrium with J firms as** $\Pi_j$. Assume that a firm will enter if it is indifferent. **Then there is SPNE with** $J^*$ **firms entering iff** $\Pi_{J^*+1} < K < \Pi_{J^*}$

**Example: Equilibrium entry with Cournot Competition. Suppose** $c(q) = cq, p(q) = a - bq, a > c \geq 0, b > 0$**. We know from before the stage 2 output level for each firm with J firms is**

$$q_J = \frac{a-c}{b(J+1)}$$

**The profit is**

$$\Pi_J = (\frac{a-c}{J+1})^2(\frac{1}{b})$$

It is easy to verify that the profit function is decreasing in J and goes to zero when J goes to infinity. Now we can solve the equilibrium number of entering firms.

$$K = (\frac{a-c}{J+1})^2(\frac{1}{b}) \Leftrightarrow J = \frac{a-c}{\sqrt{bK}} - 1$$

The number of entrant is the largest integer that is less than or equal to J. Note that as the cost of entry increases, there will be less entrants in the market and there will be more profit for the entrant in the market.

**Example: Equilibrium entry with Bertrand Competition. The market condition is the same as in Cournot. We know the profit is zero for any market with entrants more than one. If there is only one firm, then it earns the monopoly profit say** $\Pi^m$**. Of course, we need to assume that this profit is higher than entry cost. Then we know the unique SPNE must be** $J^* = 1$**.**

## 9.2 Strategic Precommitments to affect future competitions

In many of the real life example, firms can make strategic precommitments to alter the conditions of future competition. For example, firms can do R&D to reduce their marginal costs, firms can adjust their firm size to control the capacity of production. All of such things will affect the oligopoly competition in the future. We can model such features using a two-stage model as follows.

**Stage 1: Firm 1 has the option to make a commonly observed strategic investment** $k$
**Stage 2: Firm 1 and 2 engage in duopoly competition by choosing,** $s_1, s_2$**.** $\Pi_1(s_1, s_2, k), \Pi(s_1, s_2)$ **denotes the profit given** $k$**.**

Suppose there is a unique NE in stage 2 for any K, ,$(s_1^*(k), s_2^*(k))$, and assume that it is differentiable. Also assume that a higher action in one firm will reduce the reduce the other firm's profit. Therefore, in the first stage, firm 1 should induce firm 2 to lower its choice of $s_2$. The question is when can firm 1 cause firm 2 to lower $s_2$. Let $b_1(s_2, k)$ and $b_2(s_1)$ denote firm 1's and firm 2's stage 2's best response functions, we can differentiate the equilibrium condition: $s_2^* = b_2(b_1(s_2^*, k))$ to get

$$\frac{ds_2^*(k)}{dk} = \frac{b_2(s_1^*(k))}{ds_1}\left(\frac{\partial b_1(s_2^*(k), k)/\partial k}{1 - [\partial b_1(s_2^*(k), k)][db_2(s_1^*(k))/ds_1]}\right)$$

The denominator is often nonnegative which is called stability condition. $db_2/ds_1 > 0$ is called a strategic complement and negative called strategic substitute. **Example: First consider Cournot Competition. Consider firm 1's problem:**

$$max_{q_1}(a - b(q_1 + q_2) - c(k))q_1$$

**The FOC yields the best response function:**

$$a - b(q_1 + q_2) - c(k) - 2bq_1 = 0 \Leftrightarrow q_1 = \frac{a - bq_2 - c(k)}{3b}$$

**Now consider firm 2's problem:**

$$max_{q_2}(a - b(q_1 + q_2) - c_2)q_2$$

**The FOC yields the best response function:**

$$a - b(q_1 + q_2) - c_2 - 2bq_2 = 0 \Leftrightarrow q_2 = \frac{a - bq_1 - c_2}{3b}$$

**We know from previous study that the best response function is downward sloping, i.e.,** $\frac{b_2(s_1^*(k))}{ds_1} < 0$**. With higher k, it lowers the marginal cost of firm 1 which will shift its best response to the right hand side. Thus, it will produce more, i.e.** $\partial b_1(s_2^*(k), k)/\partial k > 0$**, . Hence, in this model, investment in cost reduction leads to a reduction in firm 2's output level, an effect that is benefial for firm 1.** Insert graph here.

## 10 Adverse Selection, Signaling and Screening

### 10.1 Informational asymmetries and adverse selection

**Many identical risk neutral firms**
**Cost of hiring a work: c**
**Production technology: one unit of labor produces one unit of output**
**Output price is normalized to 1.**
**N Workers differ in their productivity, $\theta$ distributed according to $F, f$**
**Firms maximize expected profits, and workers maximize maximize their utility.**
**Workers outside option $r(\theta)$**

**Given the competitive market for the firms, firms will offer a price equal to the productivity of the workers: $w^*(\theta) = \theta$.**
**Given this, the set of workers going to work is $\{\theta : r(\theta) \leq \theta\}$.**
**It is easy to verify that this outcome is Pareto optimal.**
**The sum of aggregate surplus is**

$$\int_{\underline{\theta}}^{\overline{\theta}} N[I(\theta)\theta + (1 - I(\theta)r(\theta))]dF(\theta)$$

**Obviously, it is maximized by setting $I(\theta) = 1$ if $r(\theta) \leq \theta$ and zero otherwise. w: the wage rate**
**A worker will work iff $r(\theta) \leq w$, i.e., $\Theta(w) = \{\theta : r(\theta) \leq w\}$**
**Given this, a firm's demand for labor is then**

$$z(w) = \begin{cases} 0 & if\ \mu < w \\ (0, \infty) & if\ \mu = w \\ \infty & if\ \mu > w \end{cases} \tag{6}$$

**Definition: IN the competitve labor market model with unobservable worker productivity levels, a competitive equilibrium is a wage rate $w^*$ and a set $\Theta^*$ of worker types who accept employment such that**

$$\Theta^* = \{\theta : r(\theta) \leq w^*\}$$

$$w^* = E[\theta | \theta \in \Theta^*]$$

**If we examine this equilibrium, it should not be hard to see that it is not Pareto**

**optimal. Consider the simplest case, where $r(\theta) = 0.25$ and $\theta$ is binary on $[0, 1]$ with equal probability. Then pareto optimal implies that worker with productivity 0 stay at home and 1 go to work. But now consider the competitive equilibrium. Given the final allocation is the pareto efficient allocation, the wage rate must be $w^* = 1$, however, given this wage rate, the induced outcome is all workers will go to work. Which is a contradiction. Thus pareto efficient allocation cannot be an competitive equilibrium.**

**The the next question is what kind of equilibrium could arise in the competitive market. The following is a market equilibrium. The wage is at $0.5$ and all workers accept the offer. This is indeed the case. Given the wage rate, all players will accept since there outside option is $0.25$. Given the allocation, the wage is set at the conditional expectation. What thinngs will change if $r = 0.75$? We know wage at $0.5$ and all workers stay at home is a competitive equilibrium. Given the wage rate, all worker will stay home. Given the allocation, the wage is at its expectation.**

**Adverse Selection and Market Unraveling**

$r(\theta) \leq \theta$**: it is pareto optimal for all the workers to work**

$r'(\theta) > 0$**: workers who are more productive at a firm are also more productive at home.** This assumption actualy generates adverse selection. Since the payoff of home production is greater for more capable workers, only less capable workers accept employment at any given wage $w$, i.e., $r(\theta) \leq w$.

$\theta$ **is distributed according to cdf F, f on the support** $[\underline{\theta}, \overline{\theta}]$**.**

$$w^* = E\{\theta | r(\theta) \leq w^*\}$$

This following graph can be used to illustrate the equilibrium. Insert graph here.

**Example: Consider** $r(\theta) = \alpha\theta$**,** $\alpha < 1$ **and** $\theta \sim U[0, 1]$**. Thus,** $r(0) = 0, r(\theta) < \theta, \forall\theta > 0$**. In this case** $E[\theta | r(\theta \leq w)] = E[\theta | \alpha\theta \leq w] = w/2\alpha$. Then for $\alpha > 0.5$, $E[\theta | r(\theta) \leq 0] = 0, E[\theta | r(\theta) \leq w] < w$.

Insert graph here.

Insert graph here.

**A game theoratical approach**

**The underlying structure of the market is common knowledge:** $F, r(\cdot)$
**Stage 1: two firms simultaneously announce their wage offers** there is no loss of generality to assume N firms. **Stage 2: Workers decide whether to work for a firm and which one. Ties are broken evenly.** Now we have the following proposition:

**Proposition: Let $W^*$ denote the set of competitive equilibrium wages for the adverse selection labor market model, and let $w^* = Max\{w : w \in W^*\}$.**
**(i) If $w^* > r(\underline{\theta})$ and $\exists \epsilon > 0$ such that $E[\theta | r(\theta \leq w')] > w', \forall w' \in (w^* - \epsilon, w^*)$, then $\exists$ a unique pure strategy SPNE. Emloyed workers receive a wage $w^*$, and workers with type in the set $\Theta(w^*) = \{\theta : r(\theta) \leq w^*\}$ accept employment in firms.**
**(ii) If $w^* = r(\underline{\theta})$, then there are multiple pure strategy SPNE. However, in every pure strategy SPNE, each agent's payoff exactly equals her payoff in the highest wage competitive equilibrium.**
**Proof: Let's work backwards. Given the two offers, a $\theta$ worker will work iff the highest offer is higher than $r(\theta)$. Now, we can determine firms' strategies.**
**(i) $w^* > r(\underline{\theta})$. We argue that each firm must earn zero profit. Suppose not and there is a SPNE in which a total of M workers are hired at a wage $\bar{w}$ and in which the aggregate profits of the two firms are**

$$\Pi = M(E[\theta | r(\theta) \leq \bar{w}] - \bar{w})$$

**¿0. Note that $\Pi > 0$ implies $M > 0$, which then implies $\bar{w} \geq r(\underline{\theta})$ since otherwise no worker will accept the offer. In this case, the weakly less-profitable firm, say j's profit must be no more than $\Pi/2$. But firm j can make a profit arbitrary close to $\Pi$ by offering a wage slightly higher $\bar{w}$. A contradiction. Thus, we conclude $\Pi = 0$, i.e., both firms earn zero profits.**
**Now, we argue that each firm will post a wage at $w^*$. Since we have proved that both firms are earning zero profit. Pick firm $j$, then its wage $\bar{w}_j \in W^*$ or $\bar{w} < r(\underline{\theta})$. The first case is the definition of $W^*$ and the second case is because no worker will accept the offer. Now suppose $\bar{<} w^*$, then firm j can earn strictly positive profit by offering a wage slightly lower than $w^*$. Why? Look at the graph. Here is how the condition in the proposition plays a role, it implies that the $r(\cdot)$ crosses $w^*$ from the left to right.**

**Now, we only need to prove that both firms post wage at $w^*$ plus the workers' strategies in the proposition is an equilibrium. For the firms, following the strategy yields zero profit. If he lower his wage rate, then he will have no workers and gets zero payoff. If he post a higher wage, he can not yield more than zero. Why, look at the graph.**

Insert graph here.

**(ii) $w^* = r(\underline{\theta})$, then $E[\theta | r(\theta \leq w)] < w, \forall w > w^*$. That is any firm offering wage higher than $w^*$ will incur losses. Thus, no firm will announce a wage higher than $r(\underline{\theta})$ as announcing a wage lower than that can guarantee a zero profit. Thus, the set of wage offers that can arise in SPNE is $\{(w_1, w_2); w_j \leq w^*, \forall j = 1, 2\}$. It is straightforward to verify that all the SPNE leads to the same outcome as the competitive equilibrium at wage $w^*$.**

## 10.2 Signaling

**Two types of workers: $\theta_H > \theta_L > 0$, with probability of high productivity $\lambda$**
**Before entering the job market, a worker can get some education, which is observable.**
**Education does not affect productivity, that means education is useless.**
**$c(e, \theta)$ is the cost education for type $\theta$ with education level $e$**
**$c_e(e, \theta) > 0, c_{ee}(e, \theta) > 0, c_\theta(e, \theta) < 0, c_{e\theta}(e, \theta) < 0$**

$u(w, e|\theta)$ **denote the utility of a type** $\theta$ **worker who chooses** $e$ **and receive wage** $w$**, assume** $u(w, e|\theta) = w - c(e, \theta)$

**Assume that** $r(\theta_L) = r(\theta_H) = 0$ Here is the game tree or timing of the model. Insert graph here.

**(i)The worker's strategy is optimal given the firm's strategies.**
**(ii) The belief function** $\mu(e)$ **is derived from the worker's strategy using Bayes' rule where possible. (iii) The firms' wage offers following each choice e constitute a NEof the simulaneous move wage offer game in which the probability that the worker if of high type is** $\mu(e)$**.**

Note that PBE is equivalent to SE in this setup. Why? We will only focus on pure strategy equilibrium. By convention, we first look at last information set of the game. **After seeing some education level, both firms attach a probability** $\mu(e)$ **that the worker is type** $\theta_H$**. If so, the expected productivity of the worker is** $\mu(e)\theta_H + (1 - \mu(e))\theta_L$**. In a simultaneous move wage offer game, the unique pure strategy is to offer the wage at the expected productivity.**

### 10.3 Separating equilibria

$e^*(\theta)$**: worker's equilibrium education choice**
$w^*(e^*(\theta))$**: the wage if the firms see the education level**

**Lemma 1** ***In any SPBE, we have*** $w^*(e^*(\theta_H)) = \theta_H, w^*(e^*(\theta_L)) = \theta_L$***:***

**Proof: In any PBE, beliefs on the equilibrium path must be correctly derived from the equilibrium strategy using Bayes's rule. This implies that upon seeing education level $e^*(\theta_L)(e^*(\theta_H))$, firms must assign probability one to the worker being type $\theta_L(\theta_H)$. Since we have shown that firms will both offer wage at the expected productivity, we have the result.**

**he necessary conditions are: first, no high type worker will pretend a low type. Second, no low type worker will pretend a high type.** These implies:

$$\theta_H - c(\theta_H, e_H^*) \geq \theta_L - c(\theta_H, e_L^*)$$
$$\theta_L - c(\theta_L, e_L^*) \geq \theta_H - c(\theta_L, e_H^*)$$

**Other necessary conditions are, both type can not be better off by choosing education level other than $e_H^*, e_L^*$.**

$$\theta_H - c(\theta_H, e_H^*) \geq \theta_L - c(\theta_H, e), \forall e \Leftrightarrow \theta_H - c(\theta_H, e_H^*) \geq \theta_L - c(\theta_H, 0) \Leftrightarrow c(\theta_H, e_H^*) \leq \theta_H - \theta_L$$
$$\theta_L - c(\theta_L, e_L^*) \geq \theta_L - c(\theta_L, e), \forall e \Leftrightarrow \theta_L - c(\theta_L, e_L^*) \geq \theta_L - c(\theta_L, 0) \Leftrightarrow e_L^* \leq 0$$

Thus, combine all the conditions, we have

$$c(\theta_L, e_H^*) \geq \theta_H - \theta_L \geq c(\theta_H, e_H^*), e_L^* = 0$$

**Lemma 2** *In an SPBE,* $e^*(\theta) = 0$

Insert graph here.

Insert graph here.

Insert graph here.

### 10.4 Pooing equilibrium

**The necessary and sufficient conditions are there is no incentive for both types to choose a different level of effort.** But the othere choice are off equilibrium path, so we can assign any beliefs on the information set. We know the worst case is to be perceived as a low type worker. Thus, these conditions are equilivalent to:

$$E(\theta) - c(e^*, \theta) \geq \theta_L - c(e, \theta), \forall \theta, e \Leftrightarrow E(\theta) - c(e^*, \theta_L) \geq \theta_L, \forall$$

Insert graph here.

# 11 Screening

**Two types of workers: $\theta_H > \theta_L > 0, \lambda$ is the fraction of high type**
$r(\theta_L) = r(\theta_H) = 0$
**Now, the worker can ask for a certain level of education. Education does not affect productivity, $t$.**
$u(w, t|\theta) = w - c(t, \theta)$
$c_e(e, \theta) > 0, c_{ee}(e, \theta) > 0, c_\theta(e, \theta) < 0, c_{e\theta}(e, \theta) < 0$
**Stage one: two firms simultaneously announce sets of offered contracts. A contract is a pair $(w, t)$. Each firm may announce any finite number of contracts.**
**Stage two: Given the firms' offer, workers choose the firm and contract. Assume if indifferent, then choose lower education level. If the best offer is identical, then accept each firm with half prabability.**
**Complete information case**

**Proposition: In any SPNE of the screening game with complete information, a type $\theta_i$ worker accepts contract $(w_i^*, t_i^*) = (\theta_i, 0)$. Firms earn zero profit.**
**Proof: First argue that $w_i^* = \theta_i$. Suppose not, consider $w_i^* > \theta_i$, then the firm will make a loss and could do better by not offering any contract the type $\theta_i$. Consider $w_i^* < \theta_i$. Let $\Pi > 0$ be the aggregate profits reached by the two firms on type $\theta_i$ workers. One of the firm must earn no more than $\Pi/2$ from $\theta_i$ workers. If it deviates by offering a contract $w_i^* + \epsilon, t_i^*$, it will attract all type $\theta_i$ workers. By continuity, it can earn arbitrary close to $\Pi$. A contradiction, thus, $w_i^* = \theta_i$. This also implies that in equilibrium the firms must earn zero profit.**
**Now suppose the education level is not zero. Then consider the figure, either firm can deviate to some point here and earn positive profit. The only possible point is at zero**

Insert graph here.

Insert graph here.

**Lemma: In any equilibrium, whether pooling or separating, both firms must earn zero profit.**
**Proof: Let $w_L, t_L$ and $w_H, t_H$ be the contracts chosen by the low and high type workers (could be the same). Suppose the aggregate profit is $\Pi > 0$. Then one firm must making no more than $\Pi/2$. Consider a deviation by this firm of offering $w_L + \epsilon, t_L$ and $w_H + \epsilon, t_H$. Then this firm can attract all the workers. Since $\epsilon$ can be arbitrary small and due to the continuity, this firm's profit can be arbitrary close to $\Pi$. Thus, $\Pi \leq 0$. But a firm can guarantee zero profit by not offering any contracts. Thus, $\Pi = 0$.**

**Lemma: no pooling equilibria exists.**
**Proof: Suppose there exists a pooling equilibrium at $(w^p, t^p)$. Because of the above lemma, we must have $w^p = E(\theta)$. However, given one firm is offering this offer, it is profitable for the other firm to do the following. This contract attracts all the high type but not the low type. But in this case, the firm is making positive profit.**

Insert graph here.

**Lemma: If $(w_L, t_L)$ and $(w_H, t_H)$ are the contracts for low and high abilities in a separating equilibrium, then both contracts yield zero profit: $w_L = \theta_L, w_H = \theta_H$.**
**Proof: Suppose $w_L < \theta_L$. Then either firm could earn strictly positive profit by offering a bit higher wage since at least all low type workers will accept it. If the high type also accept the contract, then the firm is even better. But this contradicts the lemma stating that both firms must earn zero profit.**
**Now suppose $w_H < \theta_H$. Look at the graph. Given the other firm is using the equilibrium strategy, the other firm can do better by instead offering $\tilde{w}, \tilde{t}$. Then all high type will accept this offer but the low type will not pretend to be the high type. Therefore such a firm can earn strictly positive profit from high type and zero profit from low type. Therefore, we must have $w_H \geq \theta_H$.**

**In conclusion,** $w_L = \theta_L$, $w_H = \theta_H$**.** Insert graph here.

**Lemma: In any separating equilibrium, the low ability workers accept contract** $\theta_L, 0$**.**
**Proof: Look at the graph. Suppose not, then the point is lying here. Then a firm can deviate to a point here. What will happen. First, the high type will not switch to this contract. Second all low type workers will accept this contract. The firm's profit is higher since wage is lower.** Insert graph here.

**Lemma: In any separating equilibrium, the high type workers accept contract** $\theta_H, \hat{t}_H$**, where** $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$**.**
**Proof: First, we know that for the low type his position is here. Then we can draw an indifference curve. We also know that the high type wage must be** $\theta_H$**. Therefore, it must lie on the right side of the low type's indifference curve. But it can only be here. Why, if it is here, then look at the high type's indifference curve. We know the new contract will attract all the high type but not the low type. But this gives the firm strictly positive profit since wage is lower than** $\theta_H$**. Therefore, it must be here. What is feature of this point? It must satisfy this equation.**

Insert graph here.

To summarize.

**Proposition: In any SPNE of the screening game, low ability worker accept contract $\theta_L, 0$ and high ability workers accept contract $(\theta_H, \hat{t}_H)$.**

Insert graph here.

# 12 Principal-Agent Problem

## 12.1 Moral hazard

$\pi \in [\underline{\pi}, \bar{\pi}]$ **profit of the object**
$e \in \{e_H, e_L\}$ **effort level or the manager's action choice, binary choice here**
$F(\pi|e), f(\pi|e)$**: the cdf and density of the profit conditional on the effort**
$F(\pi|e_H) \leq F(\pi|e_L)$**: the profit with high effort level first order stochastically dominates that with low effort. It implies that the expected profit is higher for high effort.**
**Manager:** $u(w,e), u-w > o, u_{ww} \leq 0, u(w,e_H) < u(w,e_L)$**, here assume separability** $u(w,e) = v(w) - g(e), v' > 0, v'' \leq 0, g(e_H) > g(e_L)$
**Owner: risk neutral and maximize** $E(\pi) - w$

**Optimal contract when effort is observable**

$$\max_{e \in \{e_L, e_H\}, w(\pi)} \int (\pi - w(\pi)) f(\pi|e) d\pi$$

s.t.

$$v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \overline{u}$$

**Given the effort level, the problem becomes:**

$$\min_{w(\pi)} \int w(\pi) f(\pi|e) d\pi$$

**s.t.**

$$\int v(w(\pi))f(\pi|e)d\pi - g(e)dF(\pi) \geq \overline{u}$$

**FOC:**

$$f(\pi|e) + \gamma v'(w(\pi)f(\pi|e)) = 0 \Leftrightarrow \frac{1}{v'(w(\pi))} = \gamma$$

**In the principal agent model with observable managerial effort, an optimal contract specifies that the manager choose the effort $e^*$ that maximize the above equation and pays the manager a fixed wage $w^* = v^{-1}(\bar{u} + g(e^*))$. This is the unique contract if $v'' > 0$.**

**Optimal contract when effort is not observable**

**Suppose** $v(w) = w$ we have the following proposition

**In the PA model with unobservable managerial effort and a risk neutral manager, an optimal contract generates the same effort choice and expected utilities for the manager and the owner as when effort is observable.**
**Proof: what we will do here is to consider the following contract and show it achieves the first best.**
**Consider the payment scheme $w(\pi) = \pi - \alpha$ where $\alpha$ is some constant. This is the same as selling the first to the manager. If the manager accepts this contract, he choose e to maximize his expected utility**

$$\int w(\pi)f(\pi|e)d\pi = \int \pi f(\pi|e)d\pi - \alpha - g(e)$$

**But we know this must generate the same effort level as in the first best.**
**The manager is willing to accept this contract as long as it gives him an expected utility of $\bar{u}$, i.e.,**

$$\int \pi f(\pi|e^*)d\pi - \alpha - g(e^*) \geq \bar{u}$$

**We set $\alpha$ such that the above equation holds with equality.**

$$\alpha^* = \int \pi f(\pi|e^*)d\pi - g(e^*) - \bar{u}$$

**Here we know the manager and the owner get exactly the same payoff as in the first best.**

**Risk averse manager**

$$\min_{w(\pi),e} \int w(\pi)f(\pi|e)d\pi$$

s.t.

$$\int v(w(\pi))f(\pi|e)d\pi - g(e) \geq \bar{u}$$

$$e \ solves \ \max_{\tilde{e}} \int v(w(\pi))f(\pi|\tilde{e})d\pi - g(\tilde{e})$$

Case 1: $e = e_L$.

$$\min_{w(\pi)} \int w(\pi)f(\pi|e_L)d\pi$$

s.t.

$$\int v(w(\pi))f(\pi|e_L)d\pi - g(e_L) \geq \bar{u}$$

$$\int v(w(\pi))f(\pi|e_L)d\pi - g(e_L) \geq \int v(w(\pi))f(\pi|e_H)d\pi - g(e_H)$$

Case 2: $e = e_H$.

$$\min_{w(\pi)} \int w(\pi)f(\pi|e_H)d\pi$$

s.t.

$$\int v(w(\pi))f(\pi|e_H)d\pi - g(e_H) \geq \bar{u}$$

$$\int v(w(\pi))f(\pi|e_L)d\pi - g(e_L) \leq \int v(w(\pi))f(\pi|e_H)d\pi - g(e_H)$$

$$f(\pi|e_H) + \gamma v'(w(\pi))f(\pi|e_H) + \mu[f(\pi|e_H) - f(\pi|e_L)]v'(w(\pi)) = 0$$

$$\frac{1}{v'(w(\pi))} = \gamma + \mu[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}]$$

**Lemma:** $\gamma > 0, \mu > 0$

**Proof: suppose $\gamma = 0$. Since $F(\pi|e_H)$ FOSD $F(\pi|e_L)$, then there must exist some $\pi$ such that $f(\pi|e_L)/f(\pi|e_H) > 1$. However for such $\pi$, the FOC is violated.**
**Now suppose $\mu = 0$, the the wage is a fixed rate but we know IC will be violated in this case as the agent will always want to choose low effort level.**

Now what can we say about the payment scheme. A lot. First it is not a flat rate. Consider the flat rate $\hat{w}$ such that $\frac{1}{v'(\hat{w})} = \gamma$. Then from FOC we know

$$w(\pi) > \hat{w}, \ if \ \frac{f(\pi|e_L)}{f(\pi|e_H)} < 1$$

$$w(\pi) < \hat{w}, \ if \ \frac{f(\pi|e_L)}{f(\pi|e_H)} > 1$$

**Proposition: In the principal-agent model with unobservable manager effort, a risk-averse manager, and two possible effort choices, the optimal compensation scheme for implementing $e_H$ satisfies the FOC, gives the manager expected utility $\bar{u}$, and involves a**

**larger expected wage payment than is required when effort is observable. The optimal compensation scheme for implementing $e_L$ involves the same fixed wage payment as if effort were observable. Whenever the optimal effort level with observable effort would be $e_H$, nonobservability causes a welfare loss.**