Nonparametric Identification of Bayesian Games[*]

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October 22, 2023

Abstract

This paper studies the nonparametric identification problem for Bayesian games within the private type paradigm when the researcher cannot perfectly know players' payoff structures. Under the exclusion restriction in the form of an exogenous players' participation, we show that point identification is feasible when a nonfreeness property holds; otherwise, it becomes infeasible in general and we establish partial identification with pointwise sharp nonparametric bounds. Our results can be extended to allow for corner solutions, asymmetric players, unobserved heterogeneity, and endogenous participation. As such, we have presented positive identification results and a general econometric framework for the structural analysis of general Bayesian games.

[*]We thank Xavier D'Haultfœuille for his very constructive comments and suggestions that have greatly improved our paper. We thank Gaurab Aryal, Jason Blevins, Áuero de Paula, Matthew Gentry, Emmanuel Guerre, Hiroaki Kaido, Anton Kolotilin, Lorenzo Magnolfi, Francesca Molinari, Jun Nakabayashi, Hashem Pesaran, Xun Tang, and seminar and conference participants at Australian National University, University of Technology Sydney, Stockholm School of Economics, Academia Sinica, the 2019 Conference on Advances in Econometrics, the Econometric Society 2020 World Congress, 2021 Australian Local Economic Theory Seminars, 2021 Global Virtual Seminar on Contests & Conflict, 2021 Conference on Econometrics of Games, Matching and Networks, the 5th Dongbei Econometrics Workshop, the 16th International Symposium on Econometric Theory and Applications, and the 2023 Asia-Pacific Industrial Organization Conference for their valuable comments.

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1 Introduction

Bayesian games have been one of the most important models in modern economics, as they provide a valuable framework for analyzing strategic interactions in situations where players have incomplete information, and find applications in a wide range of fields. Although incorporating incomplete information into economic models is common, analyzing Bayesian games poses challenges from a theoretical standpoint. This is due to the complexity arising from solving the equilibrium strategies and analyzing their properties. As a result, the structural analysis of Bayesian games has been relatively limited, with a notable exception being the auction model that has been extensively studied since Paarsch (1992). The reason for this exception is probably that due to the discontinuity in bidders' payoff functions, the first-order conditions in characterizing bidders' Bayesian Nash equilibrium strategies are first-order differential equations with well-specified boundary conditions. In contrast, for general Bayesian games with continuous payoffs, these conditions manifest as integral equations.

Arguably the most standard and straightforward identification problem in Bayesian games centers on the recovery of the private type distribution of players from observed actions. In this scenario, it is assumed that only one structural component is unknown to the econometrician and needs to be identified, while maintaining a known payoff structure. The nonparametric identification approach then hinges on establishing an equilibrium monotonic relationship between an observable variable and the player's private type, and this relationship is often characterized by a one-to-one mapping expressed through the first-order condition. In auctions, the nonparametric identification and estimation method developed in Guerre, Perrigne, and Vuong (2000) relies on the mapping from a bidder's private value to the equilibrium bid. Beyond auctions, only a limited number of recent papers have analyzed applications of Bayesian games with continuous payoff functions, and advanced significantly the structural analysis of Bayesian games. To name a few examples, Aryal and Gabrielli (2020) estimate a competitive nonlinear pricing model with two firms. He and Huang (2021) study nonparametric identification and estimation of the Tullock contest model with private information. Bhattacharya (2021) examines empirically R&D procurement contests, using data on research expenditures and procurement contracts. His model is of multiple stages with the third stage as a contest model. Aryal and Zincenko (2023) study identification and estimation of the Cournot competition model under incomplete information. In all these papers the only model primitive that needs to be (nonparametrically) identified is the private type distribution, while assuming a known and additive payoff function.

Building on the aforementioned literature that primarily focuses on specific applications and assumes known payoff functions, this paper addresses the nonparametric identification of general

Bayesian games within the independent private type paradigm. Departing from the existing literature, we do not impose full knowledge of the payoff function. We maintain the additive payoff structure, aligning with the common assumption in the majority of the structural work on Bayesian games with continuous payoff functions. This framework is broad enough to cover many applications such as oligopoly competition (Raith (1996)), Diamond search (Diamond (1982)), public good provision (Bergstrom, Blume, and Varian (1986)), and Tullock contest (Tullock (1980)). To tackle the identification issue, we focus on the case where a part of the (additive) payoff function is unknown to researchers. This case allows for a more general treatment compared to assuming knowledge of the entire functional form, as seen in the current state of the aforementioned literature. Assuming a known payoff function could be justified in some cases, but could be restrictive in general and can limit the model's applicability. Taking the empirical application of political campaigns in the contest model as an example, the effective cost function is not always an identity function (as the one assumed in He and Huang (2021)), if one takes into account other important factors, such as the budget constraints of the candidates or a public funding program by the government, which affects the effective cost of the candidates' expenditure. Assuming a particular functional form for this cost function could be subject to misspecification and lead to misleading results and policy recommendations. Yet little is known about whether the structural elements in the additive payoff function can be identified without imposing strong parametric, functional-form assumptions.[1] Our approach is also more practical, as it can accommodate real-world situations where complete knowledge of the functional form may not be feasible, which also makes the resulting econometric analysis more robust. Our framework allows for the adaptability to incorporate an unknown part of the payoff function into various examples, such as firms' cost functions in oligopoly competition games, searchers' cost functions in the Diamond search model, contributors' cost functions of private contribution in public good provision games, and contestants' cost of effort functions in the Tullock contest model, while remaining agnostic about the true functional form.

In order to nonparametrically identify both the private type distribution and the unknown part of the payoff function, we impose the exclusion restriction in the form of an exogenous players' participation. As a result, the private type distribution becomes independent of the number of players. Moreover, our paper builds upon the nonfreeness property, which is first introduced in D'Haultfœuille and Février (2015) to study the nonparametric identification of a triangular nonseparable model, and adapts this

1 This may explain in part why the empirical literature studying the effect of spending on electoral outcomes has largely employed reduced-form analyses. See, e.g. Jacobson (1978), Green and Krasno (1988), Levitt (1994), Gerber (1998), Erikson and Palfrey (2000), Sovey and Green (2011), and Gilens, Patterson, and Haines (2021).

property to a completely different identification problem for a general class of Bayesian games. In this paper, we first show that the model primitives are nonparametrically point identified if both the exclusion restriction and the nonfreeness property hold. Intuitively speaking, the nonfreeness property is about the observed actions and is satisfied whenever there is a crossing between any two distributions of observed actions with different numbers of players. Unlike D'Haultfœuille and Février (2015) who focus solely on establishing point identification under the nonfreeness property, our paper also explores more extensively by constructing bounds on the model primitives even when the nonfreeness property does not hold and point identification is generally unattainable. Specifically, we characterize the bounds through a two-step process, where the first step involves constructing preliminary bounds and the second step employs an iterative procedure to derive the final bounds improved upon the preliminary ones. Furthermore, we demonstrate that these final bounds are sharp in a pointwise sense. Lastly, we provide the discussion on the relationship between point and partial identification results. Importantly, we prove that the characterized bounds will collapse to a singleton whenever point identification is achieved, showcasing the robustness of our approach without requiring explicit sufficient conditions to establish point identification.

We illustrate our nonparametric identification approaches in the case of two different numbers of players. Specifically, we demonstrate both point and partial identification results in a relatively simplified manner for this particular scenario, and by utilizing a series of graphs to illustrate key concepts. Additionally, to validate the empirical efficacy of our approach, we present a numerical exercise that further substantiates the robustness of our findings. We also explore various extensions to accommodate corner solutions, asymmetric players, unobserved heterogeneity, and endogenous participation. Therefore, our results apply to an even larger class of Bayesian game models.

Our most significant contribution lies in providing nonparametric identification results for a general class of Bayesian games, as to the best of our knowledge, this paper is the first one studying nonparametric identification for general Bayesian games with continuous payoff functions. As such, it makes significant methodological contributions to the econometrics of games.[2] Moreover, our partial identification results contribute to the partial identification literature in econometrics pioneered by

2 It is worth noting that this paper deals with challenging identification problems in Bayesian games with continuous actions. While interesting work exists on the identification of Bayesian games with discrete actions, the framework in our study differs significantly, necessitating distinct identification strategies and approaches. See, e.g., Sweeting (2009), Aradillas-Lopez (2010), Bajari, Hong, Krainer, and Nekipelov (2010), Tang (2010), de Paula and Tang (2012), Wan and Xu (2014), and Lewbel and Tang (2015), among others. On the other hand, for identification of games with complete information, see, e.g., Bajari, Hong, and Ryan (2010) for identification of discrete games of complete information, and Kline (2015) for complete information games that allow generalized interaction structures and generalized behavioral assumptions.

Charles Manski; see Manski (2003) for a review of early contributions, and more recent work includes Manski and Tamer (2002), Tamer (2003), Molinari (2008), Chesher and Rosen (2013), among others, which are surveyed in Tamer (2010) and Molinari (2020). The partial identification approach has been used also in the structural analysis of auction data starting with Haile and Tamer (2003); see, e.g., Hortaçsu and McAdams (2010), Tang (2011), Aradillas-López, Gandhi, and Quint (2013), Gentry and Li (2014), and Chen, Gentry, Li, and Lu (2019), and also the papers surveyed in Li and Zheng (2021).

While the exclusion restriction in the form of an exogenous players' participation has been used in the structural auction literature, see, e.g., Athey and Haile (2002), Haile, Hong, and Shum (2003), Bajari and Hortaçsu (2005), Guerre, Perrigne, and Vuong (2009), Gentry and Li (2014), and Chen, Gentry, Li, and Lu (2019), our paper is the first one to use this restriction in addressing identification of general Bayesian games. Our paper and Guerre, Perrigne, and Vuong (2009) share a similar feature in that both try to identify two model primitives nonparametrically, namely, the private type distribution and the unknown payoff function in our case, and the private value distribution and the unknown utility function in Guerre, Perrigne, and Vuong (2009). However, our problem is significantly different from the one in Guerre, Perrigne, and Vuong (2009), because the intersection condition naturally satisfied at the boundary in auction models can be interpreted as a particular case of the nonfreeness property, and thus, point identification is always achieved. In contrast, in Bayesian games, we show that point identification is in general not attainable when the nonfreeness property is not satisfied, but partial identification with pointwise sharp bounds can be established. It is worth noting that in the recent econometrics literature, the adoption of exclusion restrictions in the form of discrete variation in certain instrumental variables has been used to address identification of nonlinear and nonseparable models in different contexts and motivated by different applications. Notably, Torgovitsky (2015) and D'Haultfœuille and Février (2015) consider triangular nonseparable models with discrete instruments, and Abbring and Ridder (2015) consider generalized accelerated failure time models with discrete covariates. They are able to attain point identification, as in Guerre, Perrigne, and Vuong (2009), because they either establish a crossing pattern (Torgovitsky (2015)) or make a large support assumption (Abbring and Ridder (2015)). Furthermore, D'Haultfœuille and Février (2020) establish partial identification of a principal-agent model, through the exogenous change in the contract structure across time, with the contract data between the French National Institute of Statistics and Economics and the interviewers the Institute hired to conduct its surveys. They obtain partial identification rather than point identification as the distributions of observed probability for conducting a survey do not cross.

Another point worth noting is that our approach relying upon the exclusion restriction is different from an alternative method that leverages multiple first-order conditions without exploiting exogenous variations. This alternative strategy often involves recovering a set of model primitives whose number is equal to the number of conditions. In contrast to our paper, which focuses on a broad class of Bayesian games with a single first-order condition, several recent papers have explored specific Bayesian models that exhibit a two-sided nature. These models allow for the derivation of multiple first-order conditions, leading to the nonparametric identification of model primitives. For instance, Luo, Perrigne, and Vuong (2018) develop a nonparametric identification approach for a nonlinear pricing model, aiming to recover the unknown utility function and private type distribution while parameterizing the cost function by leveraging the first-order conditions for both the consumer and the firm. Moreover, the benchmark principal-agent model considered in Bontemps, Lesellier, and Martimort (2022) generates two first-order conditions for both the principal and the agent, as well as the condition to ensure efficiency, which are used together to identify three unknown functions: the cost function, the surplus function, and the private type distribution. In their full model with explanatory variables and the latent heterogeneity on the demand side, the identification strategy requires excluded variables and the corresponding completeness conditions on these variables, drawing from the literature on nonlinear instrumental variable models.

Our econometric approach is based on the theoretical literature on Bayesian games. Existence and/or uniqueness of a monotone pure strategy Nash equilibrium (MPSNE) have been established in several different frameworks that complement each other using different approaches. First, Athey (2001) provides a central tool to establish the existence of MPSNE for Bayesian games that satisfies the single crossing condition.[3] Second, Van Zandt and Vives (2007) establish the existence of a greatest and a least MPSNE for Bayesian games with strategic complementarities. Polydoro (2011) further provides sufficient conditions for uniqueness. Third, Mason and Valentinyi (2010) establish the existence and uniqueness of MPSNE using the contracting mapping method. Lastly, the existence and uniqueness of MPSNE in the Tullock contest model under incomplete information has been established separately by Fey (2008), Ryvkin (2010), and Ewerhart (2014). In our paper, to accommodate as many applications as possible, we assume the existence of MPSNE, meaning that our nonparametric methodology can be applied to any of the above frameworks. We further impose strict supermodularity

3 McAdams (2003) extends Athey's model to allow partially ordered multidimensional type and action spaces. Reny (2011) further allows action spaces to be compact and locally complete metric semilattices, and type spaces to be partially ordered probability spaces.

between one's own action and type to ensure that the existing MPSNE is strictly monotone.[4]

The rest of the paper is organized as follows. Section 2 introduces the benchmark Bayesian game. Section 3 discusses the nonfreeness property and establishes the nonparametric identification results under the exclusion restriction. Section 4 illustrates the identification results using the case of two different numbers of players and presents the numerical exercise. Section 5 extends the model. Section 6 concludes. The proofs of our main results are in the Appendix.

2 Model

2.1 The Benchmark Model

We present the independent private type (IPT) Bayesian game with symmetric players in the benchmark model, and various extensions will be discussed in Section 5. There are N *ex-ante* symmetric players engaging in a Bayesian game, where $N \in \mathscr{N}$ and $\mathscr{N} \equiv \{2, 3, \cdots\}$ with the cardinality $|\mathscr{N}| = K$ and $2 \leq K < \infty$. Player $i \in \{1, \cdots, N\}$ has a private type t_i drawn from a distribution with CDF $F(\cdot|N)$ over the type space $\mathscr{T}(N) \equiv [\underline{t}(N), \bar{t}(N)] \subset \mathbb{R}$, where $F(\cdot|N)$ is absolutely continuous with an atomless density $f(\cdot|N)$. Types are drawn independently across players. All players choose actions simultaneously. Player i's action a_i is chosen from a compact action space $\mathscr{A} \subset \mathbb{R}$. Since all players are *ex-ante* symmetric, we focus on the symmetric strategy, and each player's strategy is a mapping from the type space to the action space, i.e., $s : \mathscr{T}(N) \to \mathscr{A}$. Player i's payoff depends on all players' actions and her own type, and thus is a mapping $\pi : \mathscr{A}^N \times \mathscr{T}(N) \to \mathbb{R}$. This payoff is symmetric in the actions taken by player i's rivals, thus denoted as $\pi(a_i, \mathbf{a}_{-i}, t_i)$, where $\mathbf{a}_{-i} = (\cdots, a_{i-1}, a_{i+1}, \cdots)$ denotes the action vector of all rivals of player i. The number of players N, the distribution of private types $F(\cdot|N)$, and the payoff function $\pi(\cdot,\cdot,\cdot)$ are common knowledge among all players participating in the same game.

The following assumption gives the regularity conditions and properties for the CDF $F(\cdot|N)$:

Assumption 1 *[Regularity Conditions on F] For $N \in \mathscr{N}$, let $\mathscr{F}$ be a class of functions that satisfies the following conditions, $\forall F(\cdot|N) \in \mathscr{F}$:*
(i) $F(\cdot|N)$ is a CDF with a compact support $\mathscr{T}(N) = [\underline{t}(N), \bar{t}(N)] \subset \mathbb{R}$.
(ii) $F(\cdot|N)$ admits up to $R+1$ continuous and bounded derivatives over its support with $R \geq 1$.
(iii) The PDF $f(\cdot|N)$ is bounded away from zero and infinity over $\mathscr{T}(N)$.

Assumption 1 implies that the PDF $f(\cdot|N)$ admits up to R continuous and bounded derivatives on

4 There is a large literature that establishes the existence of pure strategy equilibrium but without the property of monotonicity, to name a few, see Vives (1990), Milgrom and Weber (1985), Khan and Zhang (2014), Barelli and Duggan (2015), and He and Sun (2019).

its support $\mathscr{T}(N)$.

As discussed in the Introduction, it is usually assumed in the literature that the structure of $\pi(a_i, \mathbf{a}_{-i}, t_i)$ is additive and known by the researcher. We will follow the literature on assuming an additive payoff structure, but explore the possibility of an unknown structure in the payoff function. We impose some regularity conditions on the payoff function as follows:

Assumption 2 *[Regularity Conditions on π] For $N \in \mathscr{N}$, let Π be a class of functions that satisfy the following conditions, $\forall \pi(\cdot,\cdot,\cdot) \in \Pi$:*
(i) $\pi(a_i, \mathbf{a}_{-i}, t_i) = t_i x(a_i, \mathbf{a}_{-i}) + y(a_i, \mathbf{a}_{-i}) + z(a_i)$, where $x(\cdot,\cdot)$ and $y(\cdot,\cdot)$ are known functions of all players' actions while $z(\cdot)$ is an unknown function of player i's own action.
(ii) $x(\cdot,\cdot)$, $y(\cdot,\cdot)$, and $z(\cdot)$ admit up to $R+2$ continuous and bounded partial derivatives on their supports $\mathscr{A}^N$, $\mathscr{A}^N$, and $\mathscr{A}$ respectively, with $R \geq 1$.
(iii) $z(\cdot)$ is concave on its support $\mathscr{A}$.
(iv) $\pi(a_i, \mathbf{a}_{-i}, t_i)$ is strictly supermodular in (a_i, t_i) almost everywhere: there exist positive numbers m_x and M_x such that

$$m_x \leq \frac{\partial^2 \pi(a_i, \mathbf{a}_{-i}, t_i)}{\partial a_i \partial t_i} = \frac{\partial x(a_i, \mathbf{a}_{-i})}{\partial a_i} \leq M_x \text{ almost everywhere } \forall i.$$

Part (i) of Assumption 2 states that we focus on Bayesian games with an additive payoff structure with one unknown function $z(\cdot)$, which only depends on player i's own action a_i. Note that t_i can be replaced by a deterministic and known function of t_i such as $h(t_i)$. Assumption 2-(ii) assumes the smoothness of functions $x(\cdot,\cdot)$, $y(\cdot,\cdot)$, and $z(\cdot)$, which in turn determines the smoothness of the payoff function. Part (iii) imposes some shape restriction on the latent function $z(\cdot)$, which plays an important role when we establish the partial nonparametric identification result in Section 3.[5] Assumption 2-(iv) states that the payoff function of player i is strictly supermodular in this player's own type and action, and is essential in the next subsection's discussion on the strict monotonicity of the Bayesian Nash Equilibrium (BNE), the equilibrium notion we adopt throughout the paper.[6]

All these assumptions are standard and satisfied by the following commonly studied applications of Bayesian games.

Example 1 ***Cournot competition:*** *The private type t_i is a firm's demand characteristic and the action*

5 The characterization of bounds in Section 3.3 remains applicable if $z(\cdot)$ is a convex function; and the point identification results in Section 3.2 do not rely on this shape restriction.

6 For a Bayesian game with a strictly decreasing equilibrium strategy, we can simply redefine the private type through some strictly decreasing transformation of t_i, such as $-t_i$ or $1/t_i$.

a_i is a firm's production level.[7] *Firm i's profit is*

$$\pi(a_i, \mathbf{a}_{-i}, t_i) = \left(t_i - \sum_{j=1}^{N} a_j\right) a_i - c(a_i) = t_i a_i - a_i \sum_{j=1}^{N} a_j - c(a_i),$$

where $x(a_i, \mathbf{a}_{-i}) = a_i$, $y(a_i, \mathbf{a}_{-i}) = -a_i \sum_{j=1}^{N} a_j$, and $z(a_i) = -c(a_i)$.

Example 2 ***Diamond search model:*** *The private type t_i is a player's valuation of a match and the action a_i is a player's search intensity. The payoff function for player i is*

$$\pi(a_i, \mathbf{a}_{-i}, t_i) = t_i a_i \left(\sum_{j \neq i} a_j\right) - c(a_i),$$

where $x(a_i, \mathbf{a}_{-i}) = a_i \left(\sum_{j \neq i} a_j\right)$, $y(a_i, \mathbf{a}_{-i}) = 0$, and $z(a_i) = -c(a_i)$.

Example 3 ***Public good provision:*** *The private type t_i is a player's value of the public good, and the action a_i is the individual contribution. The payoff function for player i is*

$$\pi(a_i, \mathbf{a}_{-i}, t_i) = t_i (a_i + \sum_{j \neq i} a_j)^2 - c(a_i),$$

where $x(a_i, \mathbf{a}_{-i}) = (a_i + \sum_{j \neq i} a_j)^2$, $y(a_i, \mathbf{a}_{-i}) = 0$, and $z(a_i) = -c(a_i)$.

Example 4 ***Tullock contest:*** *The private type t_i is a player's valuation of winning the prize, and the action a_i is the expenditure in R&D contests, or funds raised in political campaigns. Then the payoff function for player i is*

$$\pi(a_i, \mathbf{a}_{-i}, t_i) = t_i \frac{a_i}{a_i + \sum_{j \neq i} a_j} - c(a_i),$$

where $\frac{a_i}{a_i + \sum_{j \neq i} a_j}$ is the Tullock contest success function, and $c(\cdot)$ is the cost function of action. This application has $x(a_i, \mathbf{a}_{-i}) = \frac{a_i}{a_i + \sum_{j \neq i} a_j}$, $y(a_i, \mathbf{a}_{-i}) = 0$, and $z(a_i) = -c(a_i)$.[8]

2.2 Strictly Monotone Pure Strategy Nash Equilibrium

In order to discuss the identification problem, we focus on the strictly monotone symmetric pure-strategy Nash Equilibrium (MPSNE) where all players adopt the same strictly increasing equilibrium strategy $s(\cdot)$. Given that all other players adopt the same equilibrium strategy, i.e., $a_j = s(t_j)$ for $j \neq i$, player i's maximization problem, when having private type t_i, can be written as:

$$\max_{a_i} \int_{\mathbf{t}_{-i} \in \mathscr{T}(N)^{N-1}} \left[t_i x(a_i, \mathbf{s}_{-i}(\mathbf{t}_{-i})) + y(a_i, \mathbf{s}_{-i}(\mathbf{t}_{-i})) + z(a_i)\right] d\mathbf{F}_{-i}(\mathbf{t}_{-i}|N), \tag{2.1}$$

7 A more standard and less general model is to assume that firms have a constant private marginal cost. However, in that model, the payoff structure is known by the researcher.

8 The payoff in a Tullock model is continuous except when all players choose zero efforts, which never happens in equilibrium (see Ewerhart (2014)). Hence, the Tullock model remains within the scope of the class of games under study.

where $\mathbf{s}_{-i}(\mathbf{t}_{-i}) = (\cdots, s(t_{i-1}), s(t_{i+1}), \cdots)$, $\mathbf{t}_{-i}$ are the types of player i's rivals, and $d\mathbf{F}_{-i}(\mathbf{t}_{-i}|N) = \cdots dF(t_{i-1}|N)dF(t_{i+1}|N)\cdots$. Therefore, the first order condition (FOC) together with the equilibrium condition leads to the following *integral equation*, where $\forall i$ and $\forall t_i$,

$$t_i \cdot \int_{\mathbf{t}_{-i}\in\mathscr{T}(N)^{N-1}} \frac{\partial x(s(t_i), \mathbf{s}_{-i}(\mathbf{t}_{-i}))}{\partial a_i} d\mathbf{F}_{-i}(\mathbf{t}_{-i}|N) + \int_{\mathbf{t}_{-i}\in\mathscr{T}(N)^{N-1}} \frac{\partial y(s(t_i), \mathbf{s}_{-i}(\mathbf{t}_{-i}))}{\partial a_i} d\mathbf{F}_{-i}(\mathbf{t}_{-i}|N) + z'[s(t_i)] = 0, \tag{2.2}$$

where the partial derivatives are with respect to the first argument of the functions $x(\cdot,\cdot)$ and $y(\cdot,\cdot)$, respectively.[9] Then we have the following result.

Proposition 1 *Under Assumptions 1 and 2, for $N \in \mathscr{N}$, the equilibrium with $s(\cdot) : \mathscr{T}(N) \to \mathscr{A}$ characterized by (2.2) for the specific class of IPT Bayesian games with the structure $[F(\cdot|N), \pi(\cdot,\cdot,\cdot)] \in \mathscr{F} \times \Pi$ satisfies the following properties:*
(i) $s(\cdot)$ is strictly increasing, and admits up to $R+1$ continuous and bounded derivative on its support $\mathscr{T}(N)$.
(ii) $s(\cdot)$ has a compact image $[\underline{a}(N), \bar{a}(N)] = [s(\underline{t}(N)), s(\bar{t}(N))] \subset \mathscr{A}$.

As discussed in the Introduction, existence and uniqueness of the monotone pure strategy Nash equilibrium have been well established in the literature. The strict supermodularity condition in Assumption 2-(iv) ensures that the equilibrium is strictly monotone. Note that the uniqueness is not essential here: with multiple equilibria, as long as players always act according to one of the equilibria, it does not affect our identification results below.

3 Nonparametric Identification under Exclusion Restriction

Suppose that the number of players N is observed, and the conditional distribution $G(\cdot|N)$ of an equilibrium action is known.[10] Econometricians are interested in identifying the underlying structure $[F(\cdot|N), z(\cdot)]$ from the observables. The identification problem asks whether $[F(\cdot|N), z(\cdot)]$ can be recovered uniquely from $[N, G(\cdot|N)]$. To this end, we impose the exclusion restriction in the form of an exogenous players' participation:

Assumption 3 *[Exclusion Restriction] The CDF $F(\cdot|\cdot)$ does not depend on the number of players N, i.e., $F(\cdot|N) = F(\cdot)$ and $f(\cdot|N) = f(\cdot)$ for all $N \in \mathscr{N}$. Both $F(\cdot)$ and $f(\cdot)$ are defined on a compact support $\mathscr{T} = [\underline{t}, \bar{t}]$.*

9 Here we focus on the interior solutions, and will consider the extension to accommodate corner solutions in Section 5.1. Further, we assume that the second order condition holds throughout the paper.

10 $G(\cdot|N)$ can be consistently estimated using common nonparametric estimation methods such as kernel and sieve.

Assumption 3 is our leading identification condition in line with the discussion in the Introduction, which assumes that the latent private type distribution is independent of the number of players. Under Assumption 3, the identification problem now amounts to establishing whether $[F(\cdot), z(\cdot)]$ is uniquely determined. It is crucial to recognize that the independence between $F(\cdot)$ and N does not extend to the equilibrium action distribution. The equilibrium strategy $s(\cdot)$ still relies on N (that is denoted as $s(\cdot|N)$ from now on), resulting in the equilibrium action distribution $G(\cdot|N)$ being dependent on N as well.

In the remainder of this section, we proceed by addressing the nonparametric identification strategy. Subsequently, we examine a sufficient condition (the nonfreeness property in Definition 2) for achieving point identification results. Finally, we present pointwise sharp bounds on the model primitives when the sufficient condition fails to hold and explore the relationship between partial and point identification outcomes.

3.1 Nonparametric Identification Strategy and Nonfreeness Property

Our identification strategy hinges on the reduction of identifying two functions, i.e., $F(\cdot)$ and $z(\cdot)$, to only one. To begin, we rewrite the FOC in (2.2). Following Proposition 1-(ii), $G(\cdot|N)$ has a compact support $[\underline{a}(N), \bar{a}(N)] = [s(\underline{t}(N)), s(\bar{t}(N))]$. For each $a \in [\underline{a}(N), \bar{a}(N)]$ and for all i, $G(a|N) = Pr(a_i \leq a|N) = Pr\left(t_i \leq s^{-1}(a)|N\right) = F(t)$, where $a = s(t)$. Hence, we can write (2.2) as:

$$t_i \cdot \int_{\mathbf{a}_{-i} \in [\underline{a}(N), \bar{a}(N)]^{N-1}} \frac{\partial x(a_i, \mathbf{a}_{-i})}{\partial a_i} d\mathbf{G}_{-i}(\mathbf{a}_{-i}|N) + \int_{\mathbf{a}_{-i} \in [\underline{a}(N), \bar{a}(N)]^{N-1}} \frac{\partial y(a_i, \mathbf{a}_{-i})}{\partial a_i} d\mathbf{G}_{-i}(\mathbf{a}_{-i}|N) + z'(a_i) = 0, \tag{3.1}$$

where $t_i = s^{-1}(a_i)$, and $\mathbf{G}_{-i}(\mathbf{a}_{-i}|N)$ denotes the joint distribution of the action profile of player i's rivals $\mathbf{a}_{-i}$ conditional on N. Alternatively, we can denote the two multiple integrals as expectations taken over $\mathbf{a}_{-i}$ and rewrite (3.1) as:

$$t_i \cdot \mathbb{E}_{\mathbf{a}_{-i}}\left[\frac{\partial x(a_i, \mathbf{a}_{-i})}{\partial a_i}\middle|N\right] + \mathbb{E}_{\mathbf{a}_{-i}}\left[\frac{\partial y(a_i, \mathbf{a}_{-i})}{\partial a_i}\middle|N\right] + z'(a_i) = 0. \tag{3.2}$$

The following proposition gives the regularity properties that the equilibrium action CDF $G(\cdot|N)$ satisfies, implied by the regularity conditions on the CDF of private types $F(\cdot)$ in Assumption 1, the payoff function $\pi(\cdot,\cdot,\cdot)$ in Assumption 2, and the equilibrium strategy function $s(\cdot)$ in Proposition 1.

Proposition 2 *Under Assumptions 1-3, the derived equilibrium action CDF $G(\cdot|N)$ for the specific class of IPT Bayesian games with the structure $[F(\cdot), \pi(\cdot,\cdot,\cdot)] \in \mathscr{F} \times \Pi$ satisfies the following conditions:*

(i) $G(\cdot|N)$ is a CDF with a compact support $[\underline{a}(N), \bar{a}(N)] = [s(\underline{t}(N)), s(\bar{t}(N))] \subset \mathscr{A}$.

(ii) $G(\cdot|N)$ admits up to $R+1$ continuous bounded derivatives on $[\underline{a}(N), \bar{a}(N)]$ with $R \geq 1$.

(iii) The PDF $g(\cdot|N)$ admits up to R continuous bounded derivatives and is bounded away from zero

and infinity on $[\underline{a}(N), \bar{a}(N)]$*, with* $R \geq 1$.

Parts (i) and (ii) of Proposition 2 are similar to parts (a) and (b) in Assumption 1. Part (iii) of Proposition 2 gives the regularity properties of the PDF $g(\cdot|N)$, which are specific to the Bayesian game model we consider, and are different from what is derived for the first-price auction model as in Guerre, Perrigne, and Vuong (2000), Guerre, Perrigne, and Vuong (2009), and Marmer and Shneyerov (2012), where the PDF of the equilibrium bids admits up to $R+1$ continuous bounded derivatives. The reason for this discrepancy arises from the nature of the first-order conditions. In general Bayesian games with continuous payoffs, they take the form of integral equations. However, in auctions, they are expressed as differential equations. Next, we summarize the conditions in Proposition 2 to define the class of action CDFs under consideration.

Definition 1 *[Regularity Conditions on G] For* $N \in \mathscr{N}$*, let* $\mathscr{G}$ *be a class of functions such that* $\forall G(\cdot|N) \in \mathscr{G}$ *satisfies all the conditions in Proposition 2.*

Let $t(\tau)$ be the τ-th quantile of $F(\cdot)$, i.e., $F\big(t(\tau)\big) = \tau$. Since $a(\tau) = s\big(t(\tau)|N\big)$, $G\big(a(\tau)|N\big) = F\big(t(\tau)\big) = \tau$, implying that $a(\tau)$ is the τ-th quantile of $G(\cdot|N)$ for each N. Therefore, (3.2) can be expressed as an equation of τ on its support $[0,1]$ as follows, known as the τ-FOC:

$$t(\tau) \cdot \mathbb{E}_{\mathbf{a}_{-N}}\left[\frac{\partial x(a(\tau), \mathbf{a}_{-N})}{\partial a}\bigg| N\right] + \mathbb{E}_{\mathbf{a}_{-N}}\left[\frac{\partial y(a(\tau), \mathbf{a}_{-N})}{\partial a}\bigg| N\right] + z'[a(\tau)] = 0, \tag{3.3}$$

where $\mathbb{E}_{\mathbf{a}_{-N}}$ denotes the expectation taken over all possible action profiles of $N-1$ rivals, and $\mathbf{a}_{-N}$ shows the action profile of these $N-1$ players. Note that we use these simplified notations by symmetry. From the τ-FOC (3.3), we also derive explicitly the inverse strategy function for interior solutions as follows:

$$\begin{aligned} t(\tau) = &- \left\{ z'[a(\tau)] + \mathbb{E}_{\mathbf{a}_{-N}}\left[\frac{\partial y(a(\tau), \mathbf{a}_{-N})}{\partial a}\bigg| N\right]\right\} \cdot \left\{\mathbb{E}_{\mathbf{a}_{-N}}\left[\frac{\partial x(a(\tau), \mathbf{a}_{-N})}{\partial a}\bigg| N\right]\right\}^{-1} \\ \equiv &\, \xi\left(a(\tau), z'[a(\tau)]\big| N\right), \end{aligned} \tag{3.4}$$

which emphasizes its dependence on the unknown z' being monotonically decreasing under Assumption 2-(iii).

To ease the notation, when N takes the value N_j, the corresponding equilibrium strategy is expressed as $s_j(\cdot) \equiv s\left(\cdot \big| N = N_j\right)$ with the inverse strategy denoted as $\xi_j(\cdot) \equiv \xi\left(\cdot, z'[\cdot]\big| N = N_j\right)$. The equilibrium action distribution is $G_j \equiv G(\cdot|N = N_j)$, where $a_j \in [\underline{a}_j, \bar{a}_j]$. The τ-FOC in (3.3) given $N = N_j$ is simplified as:

$$t(\tau) \cdot \mathbb{E}_j^x\left[a_j(\tau)\right] + \mathbb{E}_j^y\left[a_j(\tau)\right] + z'\left[a_j(\tau)\right] = 0, \tag{3.5}$$

where $\mathbb{E}_j^x\left[a_j(\tau)\right] \equiv \mathbb{E}_{\mathbf{a}_{-N_j}}\left[\frac{\partial x(a_j(\tau), \mathbf{a}_{-N_j})}{\partial a_j}\bigg| N_j\right]$ (a similar expression applies to y).

With the exogenous participation under Assumption 3, we can cancel out the common $t(\tau)$ from the τ-FOCs of different numbers of players N_i and N_j, and obtain the following result:

$$\mathbb{E}_i^y\left[a_i(\tau)\right]\mathbb{E}_j^x\left[a_j(\tau)\right]-\mathbb{E}_j^y\left[a_j(\tau)\right]\mathbb{E}_i^x\left[a_i(\tau)\right]=z'\left[a_j(\tau)\right]\mathbb{E}_i^x\left[a_i(\tau)\right]-z'\left[a_i(\tau)\right]\mathbb{E}_j^x\left[a_j(\tau)\right]. \tag{3.6}$$

Consequently, for any $(N_i,N_j)\in\mathscr{N}^2$ and any $\tau\in[0,1]$,

$$z'\left[a_j(\tau)\right]=z'\left[a_i(\tau)\right]\frac{\mathbb{E}_j^x\left[a_j(\tau)\right]}{\mathbb{E}_i^x\left[a_i(\tau)\right]}+\mathbb{E}_i^y\left[a_i(\tau)\right]\frac{\mathbb{E}_j^x\left[a_j(\tau)\right]}{\mathbb{E}_i^x\left[a_i(\tau)\right]}-\mathbb{E}_j^y\left[a_j(\tau)\right]\equiv\gamma_{ij}\left(z'\left[a_i(\tau)\right]\right), \tag{3.7}$$

where we define the operator from $z'\left[a_i(\tau)\right]$ to $z'\left[a_j(\tau)\right]$ as γ_{ij}. This operator γ_{ij} is *strictly increasing* under the strict supermodularity Assumption 2-(iv), which plays an important role in the main (partial) identification results.

Importantly, if $z'\left[a_i(\tau)\right]$ is identified, $z'\left[a_j(\tau)\right]$ is immediately identified as γ_{ij} is known. Further, the two associated action values are also linked, since $a_i(\tau)$ and $a_j(\tau)$ are the τ-th quantiles of G_i and G_j respectively:

$$a_j(\tau)=G_j^{-1}\circ G_i\big(a_i(\tau)\big)\equiv\lambda_{ij}\big(a_i(\tau)\big), \tag{3.8}$$

where we define another known operator from $a_i(\tau)$ to $a_j(\tau)$ as λ_{ij}. Therefore, for any a on G_i, $\lambda_{ij}(a)$ maps this a to a point on G_j, and accordingly, $z'\left[\lambda_{ij}(a)\right]=\gamma_{ij}\big(z'(a)\big)$.[11] Thus, $z'\left[\lambda_{ij}(a)\right]$ is identified once $z'(a)$ is determined.

When $n>0$, $\lambda_{ij}^n=\lambda_{ij}\circ\lambda_{ij}^{n-1}=\lambda_{ij}\circ\cdots\circ\lambda_{ij}$, indicating that we always map a on G_i to a point on G_j. Moreover, the composition(s) of λ_{ij} and γ_{ij} can also be used to identify z' at more points expressed as $\lambda_{ij}^n(a)$: $z'\left[\lambda_{ij}^n(a)\right]=\gamma_{ij}^n\big(z'(a)\big)$ for any $n\in\mathbb{Z}$. On the other hand, when $n<0$, $\lambda_{ij}^n=\lambda_{ij}^{-1}\circ\left(\lambda_{ij}^{-1}\right)^{n-1}=\lambda_{ij}^{-1}\circ\cdots\circ\lambda_{ij}^{-1}$. It is obvious that $\lambda_{ij}^{-1}=\lambda_{ji}$ meaning that we map a on G_j to a new a on G_i.[12] Furthermore, if $K\geq 3$, meaning the number of players N can take more than two values, more operators can be employed to identify z': indeed applying λ_{kl} where $(N_k,N_l)\in\mathscr{N}^2$ along with its composition(s) reaches more new action values; and any composition(s) in the form of $\lambda_{kl}\circ\lambda_{ij}$ where $\big(N_i,N_j,N_k,N_l\big)\in\mathscr{N}^4$ is also usable. In the end, for the purpose of nonparametrically identifying z', we consider all such operators as described above and the associated action values that can be reached starting from an arbitrary a.

Formally, we follow D'Haultfœuille and Février (2015), and collect all possible operators on a in the form of λ_{ij} defined in (3.8) along with their compositions(s). All these operators form a group denoted as Λ_a.[13] Now we are ready to introduce the concept of nonfreeness.

11 More precisely, a is a quantile of G_i, and $\lambda_{ij}(a)$ is the corresponding quantile of G_j that satisfies: $G_j\big(\lambda_{ij}(a)\big)=G_i(a)$.

12 Trivially, if $n=0$, λ_{ij}^0 is just the identity function.

13 Note that Λ_a generally depends on the starting point a.

Definition 2 *[Nonfreeness] The group Λ_a satisfies the nonfreeness property if there exists $\lambda \in \Lambda_a$ different from the identity function that admits a positive and finite number of fixed points.*

The *nonfreeness* property is first introduced by D'Haultfœuille and Février (2015). It is important to note that the nonfreeness property is about observables and can be verified from the data directly. To be more specific, it depends on the patterns of all G_js, i.e., how the exogenous number of players affects the action CDF. When $K = 2$, the nonfreeness property holds if and only if G_i and G_j cross. This is because intersection points are natural fixed points of λ_{ij} (and λ_{ji}). When $K \geq 3$, the nonfreeness property holds if at least two action CDFs cross, and might still hold even if there is no crossing among all G_js.[14]

It turns out that our identification results can be classified into two categories depending on whether the nonfreeness property holds or not. In what follows, we first show that point identification can be achieved when the nonfreeness property holds. We then consider the case when the nonfreeness property does not hold, in which case point identification is generally unattainable, and show how to establish pointwise sharp bounds.

3.2 When Nonfreeness Holds: Point Identification

Starting from an $a \in Supp(a) \equiv \cup_{j=1}^{K} \left[\underline{a}_j, \bar{a}_j\right]$, the generated *orbit* of a is denoted as $\mathscr{O}_a \equiv \{\lambda(a) | \lambda \in \Lambda_a\}$ containing all action values that can be reached if applying some operator λ to the initial a. Obviously, $\mathscr{O}_a \subset Supp(a)$. We can also define another group $\Gamma_{z'(a)}$ by collecting all possible operators on $z'(a)$ in the form of γ_{ij} defined in (3.7) along with its compositions(s). By construction, there is a bijection mapping between Λ_a and $\Gamma_{z'(a)}$.[15] For all $\tilde{a} \in \mathscr{O}_a$, there exists a $\lambda \in \Lambda_a$ such that $\tilde{a} = \lambda(a)$. Due to the bijection between Λ_a and $\Gamma_{z'(a)}$, there is exactly one $\gamma \in \Gamma_{z'(a)}$ formed in exactly the same way as λ, and more importantly, $z'(\tilde{a})$ is identified as $z'(\tilde{a}) = \gamma\big(z'(a)\big)$ up to an obvious normalization of z' at a. Indeed, if we normalize that $z'(a) = z^o$, it follows that $z'(\tilde{a}) = \gamma\big(z^o\big)$.[16] In summary, we can identify z' on the orbit $\mathscr{O}_a$, which is formally stated in the following lemma.

Lemma 1 *Under Assumptions 1-3 and the normalization that $z'(a) \equiv z^o$ for one $a \in Supp(a)$, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then for each $\tilde{a} \in \mathscr{O}_a$, z' is point identified.*

---

14 See D'Haultfœuille and Février (2015) for the detailed discussion.

15 For instance, if $\lambda \in \Lambda_a$ is defined as three compositions of λ_{ij}, i.e., λ_{ij}^3, then there is exactly one element in $\Gamma_{z'(a)}$ denoted as γ taking the form of γ_{ij}^3.

16 Normalization is a common practice in the literature on the identification of models with incomplete information, such as Luo, Perrigne, and Vuong (2018), Aryal and Gabrielli (2020), D'Haultfœuille and Février (2020), and Bontemps, Lesellier, and Martimort (2022).

Obviously, if $\mathscr{O}_a = Supp(a)$, the point identification of z' over the full support $Supp(a)$ is obtained. It turns out that this is true when the nonfreeness property holds. To see this, when the nonfreeness property holds, λ has fixed points. Suppose λ has a fixed point such that $\lambda(a^o) = a^o$ for some $a^o \in Supp(a)$, then for an arbitrary $a \in Supp(a)$ and $a \neq a^o$ we can show that a^o is the limit point of the sequence $\{\lambda^n(a)\}_{n\in\mathbb{N}}$ as n goes to infinity. Since $\{\lambda^n(a)\}_{n\in\mathbb{N}}$ is a subset of $\mathscr{O}_a$, $\mathscr{O}_a$ contains the limit point a^o. Further, due to the arbitrariness of a, inverting the above process implies that starting from a^o can lead to all points on $Supp(a)$. As a result, $Supp(a) \subset \mathscr{O}_a$ and we have that $\mathscr{O}_a = Supp(a)$. The scenario with multiple fixed points is slightly more involved, however, the fundamental reasoning remains unchanged as we can prove that $Supp(a) \subset \mathscr{O}_a$.

Once z' is identified at $\tilde{a}$, we use the inverse strategy function ξ in (3.4) to identify $F(\cdot)$ at $\tilde{t}$ such that $\tilde{t} = \xi_j(\tilde{a})$ for some N_j. More specifically, we can identify $F(\tilde{t})$ as $G_j\left(\xi_j^{-1}(\tilde{t})\right)$, because the identification of z' implies the identification of ξ_j. Consequently, we have the following point identification results.

Theorem 1 *When the nonfreeness property holds, under Assumptions 1-3 and the normalization that $z'(a) \equiv z^o$ for one $a \in Supp(a)$, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then $\mathscr{O}_a = Supp(a)$, and z' and F are point identified on $Supp(a)$ and $[\underline{t}, \bar{t}]$, respectively.*

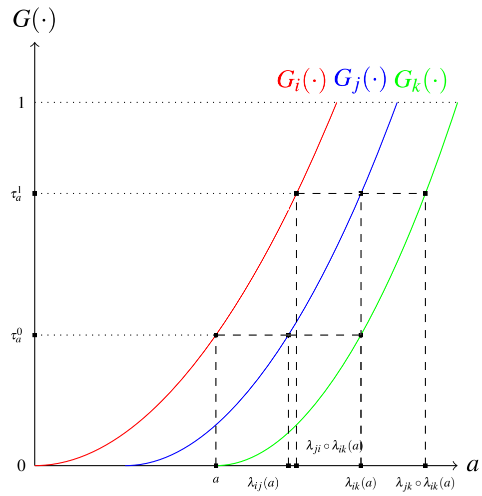
3.3 When Nonfreeness Does Not Hold: Pointwise Sharp Bounds

When the nonfreeness property does not hold, there is no fixed point for any operator $\lambda \in \Lambda_a$, and no pair of action distributions can cross. We then arrange the CDFs as $G_1, G_2, \cdots, G_K$ such that $\underline{a}_1 < \underline{a}_2 < \cdots < \underline{a}_K$ (thus, $\bar{a}_1 < \bar{a}_2 < \cdots < \bar{a}_K$). In this way, graphically we align all the CDFs within a single figure, ordering them from the left to the right in the sense that G_{j+1} stochastically dominates G_j at the first order.

Figure 1 illustrates the process of generating the orbit $\mathscr{O}_a$ when $K = 3$ and nonfreeness does not hold: applying λ_{ij} or λ_{ik} to a point a on G_i ($a \in [\underline{a}_i, \bar{a}_i]$) ends up with a new point $\lambda_{ij}(a)$ on G_j or $\lambda_{ik}(a)$ on G_k. Note that a, $\lambda_{ij}(a)$, and $\lambda_{ik}(a)$ correspond to the same τ_a^0. Then, by treating $\lambda_{ik}(a)$ as a point on G_j, applying λ_{ji} to this point gives a fourth point $\lambda_{ji} \circ \lambda_{ik}(a)$ on G_i. Applying λ_{jk} gives a fifth point $\lambda_{jk} \circ \lambda_{ik}(a)$. Again, note that $\lambda_{ji} \circ \lambda_{ik}(a)$, $\lambda_{ik}(a)$, and $\lambda_{jk} \circ \lambda_{ik}(a)$ correspond to a new τ_a^1. In principle, we can continue applying such operators and generate more points. We eventually obtain the orbit $\mathscr{O}_a$, which can always be expressed as trios consisting of the quantiles of G_i, G_j, and G_k.[17] The following lemma shows that $\mathscr{O}_a$ contains countable elements when nonfreeness does not hold.

17 This is despite the existence of some equal values as all points in this 2-dimensional coordinate system that can be projected to the same x-coordinate correspond to the same a. For instance, $\lambda_{ik}(a)$ and $\lambda_{jk} \circ \lambda_{ik}(a)$ in Figure 1.

Figure 1: Creation of orbit when $K = 3$.



Lemma 2 *When the nonfreeness property does not hold, under Assumptions 1-3, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then $\mathscr{O}_a$ is countable, $\forall a \in Supp(a)$.*

Since the support for the action $Supp(a)$ is compact, the countable orbit $\mathscr{O}_a$ is unable to completely fill it. Hence, point identification cannot be achieved in general.[18] In what follows, we turn to establish the pointwise sharp bounds on z' (and F), thereby indicating that partial identification results always hold.

By Lemma 2 and the construction of the orbit as shown in Figure 1, the countable orbit $\mathscr{O}_a$ induces a countable increasing sequence of probabilities (y-coordinates in Figure 1), denoted as $\{\tau_a^0, \tau_a^1, \cdots, \tau_a^L\}$.[19] Thus, despite the existence of equal values (see Footnote 17), we rewrite $\mathscr{O}_a$ as follows:

$$\mathscr{O}_a = \left\{a_1\left(\tau_a^0\right), \cdots, a_1\left(\tau_a^L\right);\, a_2\left(\tau_a^0\right), \cdots, a_2\left(\tau_a^L\right);\, \cdots;\, a_K\left(\tau_a^0\right), \cdots, a_K\left(\tau_a^L\right)\right\}. \tag{3.9}$$

The formulation of the orbit in this manner offers the benefit of providing a comprehensive account of all probabilities and the associated quantiles of all G_js. As a result, we will maintain the use of this expression going forward. In what follows, we first characterize preliminary bounds and then use them to derive pointwise sharp bounds.

Preliminary Bounds

To bound z', we define the normalization at $\underline{a}_1$ (the 0-th quantile of G_1) such that $z'\left(\underline{a}_1\right) \equiv z^o$. The

18 The only case in which point identification result is achieved despite the nonfulfillment of the nonfreeness property occurs when $\mathscr{O}_a$ is dense in $Supp(a)$. See Section 3.4 for detailed discussion.

19 It is important to note that this sequence generally depends on the starting point a.

orbit $\mathscr{O}_{\underline{a}_1}$ defined in (3.9) by replacing a with $\underline{a}_1$ is a proper subset of $Supp(a)$. The corresponding increasing sequence of probabilities is $\left\{\tau^0_{\underline{a}_1}, \tau^1_{\underline{a}_1}, \cdots, \tau^L_{\underline{a}_1}\right\}$, where $\tau^0_{\underline{a}_1} = 0$ by construction, and we further assume $\tau^L_{\underline{a}_1} = 1$ for the ease of illustration.[20] By Lemma 1, z' at the orbit $\mathscr{O}_{\underline{a}_1}$ is point identified:

Lemma 3 *When the nonfreeness property does not hold, under Assumptions 1-3, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then $z'\left[a_j\left(\tau^l_{\underline{a}_1}\right)\right]$ is point identified, $\forall j \in \{1, \cdots, K\}$, $\forall l \in \{0, \cdots, L\}$.*

We seek to use the point identified orbit $\mathscr{O}_{\underline{a}_1}$ to construct bounds for all other points. Suppose we start with any value that does not belong to the orbit $\mathscr{O}_{\underline{a}_1}$, say $\tilde{a}$. The orbit $\mathscr{O}_{\tilde{a}}$ defined in (3.9) by replacing a with $\tilde{a}$ is also a proper subset of $Supp(a)$. The corresponding increasing sequence of probabilities is $\left\{\tau^0_{\tilde{a}}, \cdots, \tau^{L-1}_{\tilde{a}}\right\}$. The following lemma reveals an important relationship between the two increasing sequences of probabilities:

Lemma 4 *When the nonfreeness property does not hold, under Assumptions 1-3, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then $\tau^l_{\underline{a}_1} < \tau^l_{\tilde{a}} < \tau^{l+1}_{\underline{a}_1}$, $\forall j \in \{1, \cdots, K\}$, $\forall l \in \{0, \cdots, L-1\}$.*

By Lemma 4 and the shape restriction in Assumption 2-(iii), we can bound z' for each point in $\mathscr{O}_{\tilde{a}}$ in the following manner:

$$
\begin{aligned}
& \tau^l_{\underline{a}_1} < \tau^l_{\tilde{a}} < \tau^{l+1}_{\underline{a}_1}, \\
\Leftrightarrow \quad & a_j\left(\tau^l_{\underline{a}_1}\right) < a_j\left(\tau^l_{\tilde{a}}\right) < a_j\left(\tau^{l+1}_{\underline{a}_1}\right), \\
\Leftrightarrow \quad & z'\left[a_j\left(\tau^{l+1}_{\underline{a}_1}\right)\right] \leq z'\left[a_j\left(\tau^l_{\tilde{a}}\right)\right] \leq z'\left[a_j\left(\tau^l_{\underline{a}_1}\right)\right].
\end{aligned}
$$

This is summarized in the following lemma.

Lemma 5 *When the nonfreeness property does not hold, under Assumptions 1-3, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then $z'\left[a_j\left(\tau^l_{\tilde{a}}\right)\right] \in \left[z'\left[a_j\left(\tau^{l+1}_{\underline{a}_1}\right)\right], z'\left[a_j\left(\tau^l_{\underline{a}_1}\right)\right]\right]$, $\forall j \in \{1, \cdots, K\}$, $\forall l \in \{0, \cdots, L-1\}$*

Recall that associated with the group $\Lambda_{\tilde{a}}$ that generates $\mathscr{O}_{\tilde{a}}$, there is another group $\Gamma_{z'(\tilde{a})}$ as introduced in the beginning of Section 3.2 whose element/operator can always relate $z'\left[a_j(\tau^l_{\tilde{a}})\right]$ to $z'(\tilde{a})$. Thus, we let $z'(\tilde{a}) = \phi^{j,l}\left(z'\left[a_j(\tau^l_{\tilde{a}})\right]\right)$, where the inverse of $\phi^{j,l}$ belongs to $\Gamma_{z'(\tilde{a})}$. Another noteworthy observation is that $\phi^{j,l}$ is strictly increasing, as it builds on the composition(s) of γ_{ij} which is always

20 Whether $\tau^L_{\underline{a}_1} = 1$ holds depends on the shapes of all G_js. While the case of $\tau^L_{\underline{a}_1} < 1$ adds complexity to the characterization of preliminary bounds in terms of expression, it does not introduce any new fundamental concept. It simply requires the consideration of one-sided bounds for the interval $(\tau_{\underline{a}_1}, 1]$.

strictly increasing under Assumption 2-(iv). Consequently, for each j and l, we derive one pair of bounds on $z'(\tilde{a})$ as

$$z'(\tilde{a}) \in \left[\phi^{j,l}\left(z'\left[a_j(\tau_{\underline{a}_1}^{l+1})\right]\right), \phi^{j,l}\left(z'\left[a_j(\tau_{\underline{a}_1}^{l})\right]\right)\right].$$

We then collect all such bounds by varying j and l to derive the tightest pair as the preliminary bounds on $z'(\tilde{a})$.

Proposition 3 *When the nonfreeness property does not hold, under Assumptions 1-3 and the normalization that $z'(\underline{a}_1) \equiv z^o$, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then $z'(\cdot)$ is nonparametrically partially identified on the supports $Supp(a) \setminus \mathscr{O}_{\underline{a}_1}$. Specifically, for $\tilde{a} \in Supp(a) \setminus \mathscr{O}_{\underline{a}_1}$, we have $z'(\tilde{a}) \in \left[L(z'(\tilde{a}); z^o), U(z'(\tilde{a}); z^o)\right]$ where*

$$L\left(z'(\tilde{a}); z^o\right) \equiv \sup_{\substack{j\in\{1,\cdots,K\},\\ l\in\{0,\cdots,L-1\}}} \left\{\phi^{j,l}\left(z'\left[a_j\left(\tau_{\underline{a}_1}^{l+1}\right)\right]\right)\right\}; \tag{3.10}$$

$$U\left(z'(\tilde{a}); z^o\right) \equiv \inf_{\substack{j\in\{1,\cdots,K\},\\ l\in\{0,\cdots,L-1\}}} \left\{\phi^{j,l}\left(z'\left[a_j\left(\tau_{\underline{a}_1}^{l}\right)\right]\right)\right\}. \tag{3.11}$$

Since the orbit $\mathscr{O}_{\underline{a}_1}$ depends on the initial point $\underline{a}_1$, at which z' is normalized, the bounds are functions of z^o. It is useful to point out that for each j and l, $z'\left[a_j\left(\tau_{\underline{a}_1}^{l}\right)\right]$ is a strictly increasing function of z^o under Assumption 2-(iv). Note that when the CDFs with different N_js are far away from each other, the orbit $\mathscr{O}_{\underline{a}_1}$ may be finite, thus the associated bounds are also finite. Thus, the tightest pair can be taken as the maximum of all lower limits and the minimum of all upper limits. In contrast, when the CDFs are close to each other, the orbit $\mathscr{O}_{\underline{a}_1}$ can become infinite. Consequently, we can derive an infinite number of bounds and identify the most stringent ones by computing the infimum and supremum. In this case, these preliminary bounds might be relatively narrow and informative even without point identification results.

Pointwise Sharp Bounds

We now delve into the discussion of how the preliminary bounds can be refined. Again, we focus on the target $\tilde{a}$. For the subsequent analysis, we assume it to be the $\tau_{\tilde{a}}^{\tilde{l}}$-th quantile of some $G_{\tilde{j}}$.[21] When establishing the preliminary bounds for $z'(\tilde{a})$, we only use the point identified z' on the orbit $\mathscr{O}_{\underline{a}_1}$. However, more information is available: consider the interval $\left(\tau_{\underline{a}_1}^{\tilde{l}}, \tau_{\underline{a}_1}^{\tilde{l}+1}\right)$ that contains $\tau_{\tilde{a}}^{\tilde{l}}$.[22] Although at the two endpoints, $z'\left[a_{\tilde{j}}(\cdot)\right]$ is point identified following Lemma 5, these two endpoints are distant

21 If $\tilde{a}$ falls into the support of multiple CDFs, pick anyone. See Footnote 17.

22 Note that we only need to consider the interval $\left(\tau_{\underline{a}_1}^{\tilde{l}}, \tau_{\underline{a}_1}^{\tilde{l}+1}\right)$, since the construction of preliminary bounds already accounts for other intervals such as $\left(\tau_{\underline{a}_1}^{l}, \tau_{\underline{a}_1}^{l+1}\right)$ for a general l.

from $\tau_{\tilde{a}}^{\tilde{l}}$. In contrast, for $\tau \in \left(\tau_{\underline{a}_1}^{\tilde{l}}, \tau_{\underline{a}_1}^{\tilde{l}+1}\right)$ but $\tau \neq \tau_{\tilde{a}}^{\tilde{l}}$, while $z'\left[a_{\tilde{j}}(\tau)\right]$ is only partially identified, this τ is closer to $\tau_{\tilde{a}}^{\tilde{l}}$ relative to $\tau_{\underline{a}_1}^{\tilde{l}}$.

Specifically, from the preliminary bounds in Proposition 3, we can bound $z'\left[a_{\tilde{j}}(\tau)\right]$ as:

$$z'\left[a_{\tilde{j}}(\tau)\right] \in \left[L\left(z'\left[a_{\tilde{j}}(\tau)\right];z^o\right),\ \ U\left(z'\left[a_{\tilde{j}}(\tau)\right];z^o\right)\right]. \tag{3.12}$$

Next, if we change the initial point from $\underline{a}_1$ to be $a_{\tilde{j}}(\tau)$, we obtain a different set of lower and upper bounds on the target $z'(\tilde{a})$ as:

$$z'(\tilde{a}) \in \left[L\left(z'(\tilde{a});z'\left[a_{\tilde{j}}(\tau)\right]\right),\ \ U\left(z'(\tilde{a});z'\left[a_{\tilde{j}}(\tau)\right]\right)\right].$$

However, in the above bounds, $z'\left[a_{\tilde{j}}(\tau)\right]$ is only partially identified as in (3.12). Combined with the observation that the preliminary lower and upper bounds from the formulas in (3.10) and (3.11) are strictly increasing in z^o, we can bound $z'(\tilde{a})$ in an alternative way through inequality scaling:

$$z'(\tilde{a}) \in \left[L\left(z'(\tilde{a});L\left(z'\left[a_{\tilde{j}}(\tau)\right];z^o\right)\right), U\left(z'(\tilde{a});U\left(z'\left[a_{\tilde{j}}(\tau)\right];z^o\right)\right)\right]. \tag{3.13}$$

Actually for each $\tau \in \left(\tau_{\underline{a}_1}^{\tilde{l}}, \tau_{\underline{a}_1}^{\tilde{l}+1}\right) \setminus \left\{\tau_{\tilde{a}}^{\tilde{l}}\right\}$, a new pair of bounds in the form of (3.13) can be established for $z'(\tilde{a})$. Due to the arbitrariness of τ, pooling all such bounds generated from each τ together can potentially improve the bounds for $z'(\tilde{a})$ when compared to the preliminary bounds.

Moreover, since $z'\left[a_{\tilde{j}}(\tau)\right]$'s bounds can also be improved by employing the already refined bounds on $z'(\tilde{a})$ in a similar fashion, this process can iterate until convergence happens, which are characterized by fixed points in functional analysis:

$$LL\left(z'(\tilde{a});z^o\right) = \sup_{\substack{\tau \in \left[\tau_{\underline{a}_1}^{\tilde{l}}, \tau_{\underline{a}_1}^{\tilde{l}+1}\right], \\ \tau \neq \tau_{\tilde{a}}^{\tilde{l}}}} \left\{L\left(z'(\tilde{a});LL\left(z'\left[a_{\tilde{j}}(\tau)\right];z^o\right)\right)\right\}; \tag{3.14}$$

and

$$UU\left(z'(\tilde{a});z^o\right) = \inf_{\substack{\tau \in \left[\tau_{\underline{a}_1}^{\tilde{l}}, \tau_{\underline{a}_1}^{\tilde{l}+1}\right], \\ \tau \neq \tau_{\tilde{a}}^{\tilde{l}}}} \left\{U\left(z'(\tilde{a});UU\left(z'\left[a_{\tilde{j}}(\tau)\right];z^o\right)\right)\right\}, \tag{3.15}$$

where $LL(\cdot;\cdot)$ and $UU(\cdot;\cdot)$ denote the final bound functions. These two final bound functions can be derived from the following two fixed-point problems, leveraging the monotonicity of the quantile function $a_{\tilde{j}}(\cdot)$ and the model primitive $z'(\cdot)$:

$$LL(x;z^o) = \sup_{\substack{y \in [\underline{z},\bar{z}], \\ y \neq x}} \left\{L\left(x;LL(y;z^o)\right)\right\}; \tag{3.16}$$

and

$$UU(x;z^o)=\inf_{\substack{y\in[\underline{z},\bar{z}],\\ y\neq x}}\left\{U\left(x;UU(y;z^o)\right)\right\},\tag{3.17}$$

where $\underline{z}\equiv z'\left[a_{\tilde{j}}\left(\tau_{\underline{a}_1}^{\tilde{l}+1}\right)\right]$ and $\bar{z}\equiv z'\left[a_{\tilde{j}}\left(\tau_{\underline{a}_1}^{\tilde{l}}\right)\right]$. Note that the domain of the two functions $[\underline{z},\bar{z}]$ is also the image of the two functions. In addition, we have $LL(\bar{z};z^o)=UU(\bar{z};z^o)$ and $LL(\underline{z};z^o)=UU(\underline{z};z^o)$, since z' is point identified at the two endpoints, implying the compactness of the functional space. The next lemma states the existence of the two functions $LL(\cdot;z^o)$ and $UU(\cdot;z^o)$ via the Schauder fixed-point theorem in functional analysis (see Schauder (1930) and Bonsall (1962)).

Lemma 6 *When the nonfreeness property does not hold, under Assumptions 1-3 and the normalization that $z'(\tilde{a})\equiv z^o$, consider an action distribution $G(\cdot|\cdot)\in\mathscr{G}$. Then the two functions $LL(\cdot;z^o)$ and $UU(\cdot;z^o)$ defined implicitly in the two functional fixed-point problems (3.16) and (3.17) have at least one solution.*

Lemma 6 implies that for any $\tilde{a}\in Supp(a)\setminus\mathscr{O}_{\underline{a}_1}$, there exist at least one lower and one upper bounds for $z'(\tilde{a})$. Furthermore, these bounds, which utilize the information beyond the identification result on the orbit $\mathscr{O}_{\underline{a}_1}$, are weakly tighter than those in Proposition 3. If multiple solutions exist, we can always select the pair that produces the tightest bounds by taking the upper envelope of all final lower bounds and the lower envelope of all final upper bounds.

The bounds on z' implies those on F. This is because for each $\tilde{j}\in\{1,\cdots,K\}$ the inverse strategy function $\xi_{\tilde{j}}(\cdot)\equiv\xi\left(a,z'(a)\middle|N_{\tilde{j}}\right)$ in (3.4) is strictly decreasing in $z'(a)$ for a fixed a under Assumption 2-(iv). Hence, the bounds on $z'(\tilde{a})$ translate to those on $\xi_{\tilde{j}}^{-1}(\tilde{t})$ using monotonicity arguments, such that $s_{\tilde{j}}(\tilde{t})=\tilde{a}$ (thus $\xi_{\tilde{j}}(\tilde{a})=\tilde{t}$) and $\tilde{a}$ is one equilibrium action with the number of players being $N_{\tilde{j}}$. Since $G_{\tilde{j}}$ is also strictly increasing by Proposition 2-(iii), for this $\tilde{j}$, we derive one pair of bounds on $F(\tilde{t})=G_{\tilde{j}}\left(\xi_{\tilde{j}}^{-1}(\tilde{t})\right)$. Importantly, under the exclusion restriction (Assumption 3), the above argument holds for each $\tilde{j}\in\{1,\cdots,K\}$, and finally, we pool all bounds generated from each $\tilde{j}$ together to further refine the bounds on $F(\tilde{t})$.

As a result, we have the following main theorem that gives the final bounds of z' and F on the rest of their respective supports: $Supp(a)\setminus\mathscr{O}_{\underline{a}_1}$ and $[\underline{t},\bar{t}]\setminus\{t(\tau_0)),\cdots,t(\tau_L)\}$. This theorem also shows that these are the pointwise sharp bounds on z' at any fixed $\tilde{a}$ and on F at any fixed $\tilde{t}$.

Theorem 2 *When the nonfreeness property does not hold, under Assumptions 1-3 and the normalization that $z'(\underline{a}_1)\equiv z^o$, consider an action distribution $G(\cdot|\cdot)\in\mathscr{G}$. Then $z'(\cdot)$ and $F(\cdot)$ are nonparametrically partially identified on the supports $Supp(a)\setminus\mathscr{O}_{\underline{a}_1}$ and $[\underline{t},\bar{t}]\setminus\{t(\tau_0)),\cdots,t(\tau_L)\}$, respectively. Specifically,* *(i) for $\tilde{a} \in Supp(a) \setminus \mathscr{O}_{\underline{a}_1}$, let $\tilde{a} = a_{\tilde{j}}\left(\tau_{\tilde{a}}^{\tilde{l}}\right)$. If $\tau_{\tilde{a}}^{\tilde{l}} \in \left(\tau_{\underline{a}_1}^{\tilde{l}}, \tau_{\underline{a}_1}^{\tilde{l}+1}\right)$, denote the classes of solutions to the two problems defined in (3.16) and (3.17) as $\mathscr{L}$ and $\mathscr{U}$, respectively. Then $z'(\tilde{a}) \in \left[LB_{z'(\tilde{a})}, UB_{z'(\tilde{a})}\right]$, where*

$$LB_{z'(\tilde{a})} \equiv \sup\left\{LL^*\left(z'(\tilde{a});z^o\right) : LL^* \in \mathscr{L}\right\}; \tag{3.18}$$

$$UB_{z'(\tilde{a})} \equiv \inf\left\{UU^*\left(z'(\tilde{a});z^o\right) : UU^* \in \mathscr{U}\right\}. \tag{3.19}$$

Moreover,

(ii) for $\tilde{t} \in [\underline{t}, \bar{t}] \setminus \{t(\tau_0), \cdots, t(\tau_L)\}$, we have $F(\tilde{t}) \in \left[LB_{F(\tilde{t})}, UB_{F(\tilde{t})}\right]$, where

$$LB_{F(\tilde{t})} \equiv \max_{\tilde{j}\in\{1,\cdots,K\}} G_{\tilde{j}}\left(LB_{\xi_{\tilde{j}}^{-1}(\tilde{t})}\right);$$

$$UB_{F(\tilde{t})} \equiv \min_{\tilde{j}\in\{1,\cdots,K\}} G_{\tilde{j}}\left(UB_{\xi_{\tilde{j}}^{-1}(\tilde{t})}\right),$$

where for $\tilde{j} \in \{1, \cdots, K\}$,

$$LB_{\xi_{\tilde{j}}^{-1}(\tilde{t})} \equiv \left(\xi_{\tilde{j}}^{*}\right)^{-1}\left(\xi_{\tilde{j}}^{-1}(\tilde{t}), UB_{z'\left[\xi_{\tilde{j}}^{-1}(\tilde{t})\right]}\right);$$

$$UB_{\xi_{\tilde{j}}^{-1}(\tilde{t})} \equiv \left(\xi_{\tilde{j}}^{*}\right)^{-1}\left(\xi_{\tilde{j}}^{-1}(\tilde{t}), LB_{z'\left[\xi_{\tilde{j}}^{-1}(\tilde{t})\right]}\right).$$

Finally, these bounds are sharp.

Theorem 2 provides the best nonparametric bounds on the model primitives z' and F. Take z' as an example, a lower (upper) bound is called *pointwise sharp* if for all $\tilde{a}$ that is partially identified, we can always construct a function $\hat{z}'$ such that $\hat{z}'(\tilde{a})$ is arbitrarily close to this lower (upper) bound and all the restrictions given by the data and the model hold (see the definition in Nelson, Molina, Lallena, and Flores (2004), Tankov (2011), and Bartl, Kupper, Lux, Papapantoleon, and Eckstein (2022)). The proof showing the pointwise sharpness of these bounds follows this definition and is similar to that of Theorem 3.2 in D'Haultfœuille and Février (2020) but more involved. This is because the bounds derived in their Theorem 3.2, which mirrors our preliminary bounds in Proposition 3, cannot be further refined, as these bounds are the same for all the points that are partially identified, and thus are pointwise sharp. In contrast, our iterative procedure can in general improve the preliminary bounds by generating weakly tighter bounds.

3.4 Discussion of Nonparametric Identification Results

This subsection serves as a concluding remark of Section 3. Specifically, we aim to discuss when the point or partial identification is achieved based on the results of the previous subsections.

First of all, the nonfreeness property in Definition 2 is a *sufficient* condition for the point identification results of the model primitives. An important question to consider is whether point identification is achievable when the nonfreeness property fails, and this is answered in the following proposition.

Proposition 4 *When the nonfreeness property does not hold, under Assumptions 1-3, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. Then $\forall a \in Supp(a)$,*
(i) if $\mathscr{O}_a$ is discrete in $Supp(a)$, z' and F are partially identified on $Supp(a)$ and $[\underline{t}, \bar{t}]$, respectively;
(ii) if $\mathscr{O}_a$ is dense in $Supp(a)$, z' and F are point identified on $Supp(a)$ and $[\underline{t}, \bar{t}]$, respectively.

Proposition 4 distinguishes between point and partial identification based on the distribution of $\mathscr{O}_a$ within its superset $Supp(a)$, when the nonfreeness property is not satisfied and $\mathscr{O}_a \subset Supp(a)$ under Lemma 2.[23] When $\mathscr{O}_a$ is dense in $Supp(a)$, any $\tilde{a} \in Supp(a)$ either belongs to $\mathscr{O}_a$ or is a limit point of a sequence $\{a_n\} \subset \mathscr{O}_a$ such that $\lim_{n\to\infty} a_n = \tilde{a}$. In the former case, the point identification of z' at $\tilde{a}$ follows Lemma 1, and in the latter case, $z'(\tilde{a}) = \lim_{n\to\infty} z'(a_n)$ is also point identified due to the continuity of $z'(\cdot)$.

Proposition 4 is high level. Theoretically, distinguishing between cases (i) and (ii) is a difficult task that typically demands a thorough analysis of the specific model at hand. D'Haultfœuille and Février (2015) are able to establish the point identification of a triangular nonseparable model if the nonfreeness property fails under their Assumption 4 (regularity and nonperiodicity). They show that the nonperiodicity condition (which requires $K \geq 3$) can be treated as a rank condition, under which the orbit $\mathscr{O}_a$ of any a is dense in $Supp(a)$ with the aid of Hölder's and Denjoy's theorems in group and dynamic systems.[24] However, the nonperiodicity condition is not sufficient for point identification in our model. Specifically, the support of the equilibrium action a varies with the number of players N in our model (see Proposition 1-(ii) and Definition 1-(iii)), while the support of the endogenous variable X (analogous to our equilibrium action a) is independent of the instrument Z (analogous to our exogenous number of players N) denoted as $[\underline{x}, \bar{x}]$ with $-\infty \leq \underline{x} < \bar{x} \leq \infty$ in their model (see Assumption 3-(i) in D'Haultfœuille and Février (2015)).

Instead of seeking sufficient conditions for the density of $\mathscr{O}_a$ as in D'Haultfœuille and Février (2015), we adopt an alternative approach. In Section 3.3, we characterize sharp bounds on a pointwise base when nonfreeness does not hold. The subsequent corollary demonstrates that when $\mathscr{O}_a$ is dense in $Supp(a)$ (thus case (iii) in Proposition 4), the preliminary bounds in Proposition 3 collapse to a

23 Proposition 4 covers all possible scenarios, as it is not feasible for $\mathscr{O}_a$ to be discrete in one part and dense in another. This is due to the fact that the density of a subset of $\mathscr{O}_a$ will propagate throughout the entire orbit based on its construction. Consequently, $\mathscr{O}_a$ can only be either discrete or dense within $Supp(a)$.

24 Note that the models in this paper and D'Haultfœuille and Février (2015) are drastically different: we consider a general Bayesian game, while they study a triangular nonseparable model.

singleton.

Corollary 1 *When the nonfreeness property does not hold, under Assumptions 1-3 and the normalization that $z'(\tilde{a}) = z^o$, consider an action distribution $G(\cdot|\cdot) \in \mathscr{G}$. If further $\mathscr{O}_{\underline{a}_1}$ is dense in $Supp(a)$, then for $\tilde{a} \in Supp(a) \setminus \mathscr{O}_{\underline{a}_1}$, the bounds in Proposition 3 reduce to a singleton:*

$$L\left(z'(\tilde{a}); z^o\right) = U\left(z'(\tilde{a}); z^o\right),$$

and thus, z' is point identified on $Supp(a)$. Further, F is also point identified on $[\underline{t}, \bar{t}]$.

Corollary 1 carries great significance as it illustrates that the preliminary (and the pointwise sharp) bounds are valid regardless of whether $\mathscr{O}_{\underline{a}_1}$ is distributed densely or discretely within $Supp(a)$. This highlights the advantage of our bounds, as they can achieve point identification results without relying on *ex-ante* sufficient conditions. This flexibility is particularly valuable as it circumvents the challenging task of determining such conditions and allows us to obtain meaningful estimates even in situations where the density of $\mathscr{O}_{\underline{a}_1}$ is not precisely known.

4 Illustration when $K = 2$

We have so far established partial and point identification results of model primitives (z' and F) for a general K. In this section, we apply these identification approaches to the case when $K = 2$, which sheds light on the numerical illustration in subsection 4.2. To simplify the notation, suppose N takes two values N_1 and N_2, with the corresponding action CDFs G_1 and G_2 respectively. The associated support of a is $Supp(a) \equiv [\underline{a}_1, \bar{a}_1] \cup [\underline{a}_2, \bar{a}_2]$.

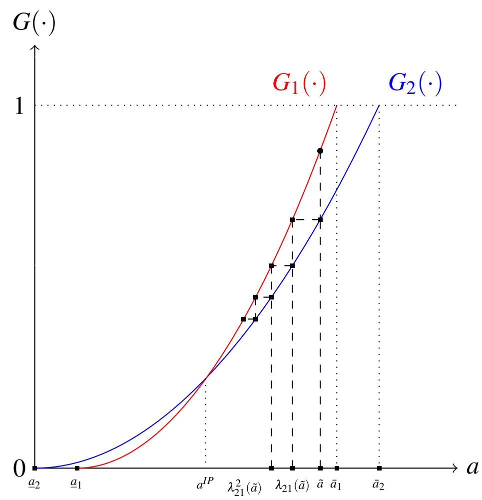
4.1 Point and Partial Identification when $K = 2$

The discussion after Definition 2 shows that whether we have point or partial identification when $K = 2$ depends completely on whether G_1 and G_2 cross or not. In the following, we first illustrate the point identification of z' when G_1 and G_2 cross with each other. We focus on the case when they cross once, as the case with multiple crossings is a straightforward adaption of this one-crossing case.

We assume G_1 and G_2 cross once as in Figure 2, thus satisfying the nonfreeness property, as the crossing point is the fixed point of λ_{12}: $\lambda_{12}(a^{IP}) = a^{IP}$.

For ease of exposition, we normalize z' at a^{IP}: $z'\left(a^{IP}\right) \equiv z^o$. Then, for any target $\tilde{a} \in \left(a^{IP}, \max\{\bar{a}_1, \bar{a}_2\}\right)$, the sequence $\left\{\lambda_{21}^n(\tilde{a})\right\}_{n\in\mathbb{N}}$ is decreasing and converges to a^{IP}. Hence, the function value z' at $\tilde{a}$ can be point identified as $z'(\tilde{a}) = \lim_{n\to\infty} \gamma_{21}^n(z^o)$, which holds by the continuity of z' in Assumption 2-(ii). Similarly, for any target $\tilde{a} \in \left(\min\{\underline{a}_1, \underline{a}_2\}, a^{IP}\right)$, the sequence $\left\{\lambda_{12}^n(\tilde{a})\right\}_{n\in\mathbb{N}}$ is increasing and converges to a^{IP}. Hence, the function value z' at $\tilde{a}$ can be point identified as

Figure 2: Point identification if G_1 and G_2 cross once.



$z'(\tilde{a}) = \lim_{n\to\infty} \gamma_{12}^n(z^o)$, which holds by the continuity of z' in Assumption 2-(ii). Consequently, we point identify $z'(\tilde{a})$ for all $\tilde{a} \in Supp(a)$. In addition, the point identification of z' leads to that of F following Theorem 1.

Next, we briefly address how the point identification results can be obtained if G_1 and G_2 intersect multiple times at $\{a^1, \cdots, a^M\}$, where $a^1 \leq \cdots \leq a^M$. Again, for ease of exposition, we normalize z' at a^M without loss of generality. These fixed points partition $Supp(a)$ into non-overlapping regions. For one target $\tilde{a}$ from the top region denoted as $\left(a^M, \max\{\bar{a}_1, \bar{a}_2\}\right)$, we can always point identify $z'(\tilde{a})$ by the same approach as shown in Figure 2. Then we move to the next region $\left(a^{M-1}, a^M\right)$, since $z'(\tilde{a})$ at each $\tilde{a}$ from this region is also point identified as above by constructing a sequence increasingly converging to a^M, we can also construct another sequence decreasingly converging to a^{M-1} from this $\tilde{a}$ and point identify a^{M-1} by the continuity of z'. Repeating this process, we can further identify z' at any $\tilde{a}$ that is not a fixed point along with all fixed points, thus achieving point identification of z' on $Supp(a)$. Again, the point identification of z' leads to that of F. One thing worth noting for this multiple-crossing case is that the sequence from one $\tilde{a}$ and converges to a fixed point is not necessarily unique as one can choose a different fixed point as the limit, but all such sequences must uniquely define $z'(\tilde{a})$. See the discussion in Guerre, Perrigne, and Vuong (2009).

Subsequently, we turn to the discussion of the partial identification. No crossing of G_1 and G_2 is equivalent to G_2 stochastically dominating G_1 at the first-order or vice versa, thus the nonfreeness property does not hold, and the orbit $\mathscr{O}_a$ for any $a \in Supp(a)$ is finite. When $K = 2$, it is also notable that this orbit is a monotone sequence as shown in Figure 3. Therefore, we can simplify the characterization of the bounds on z', which differs slightly from the approach used in Section

3.3. Without loss of generality, let us assume G_2 first-order stochastically dominates G_1. Give the normalization that $z'(\underline{a}_1) \equiv z^o$, the orbit $\mathscr{O}_{\underline{a}_1}$ is shown in the following Figure 3a:

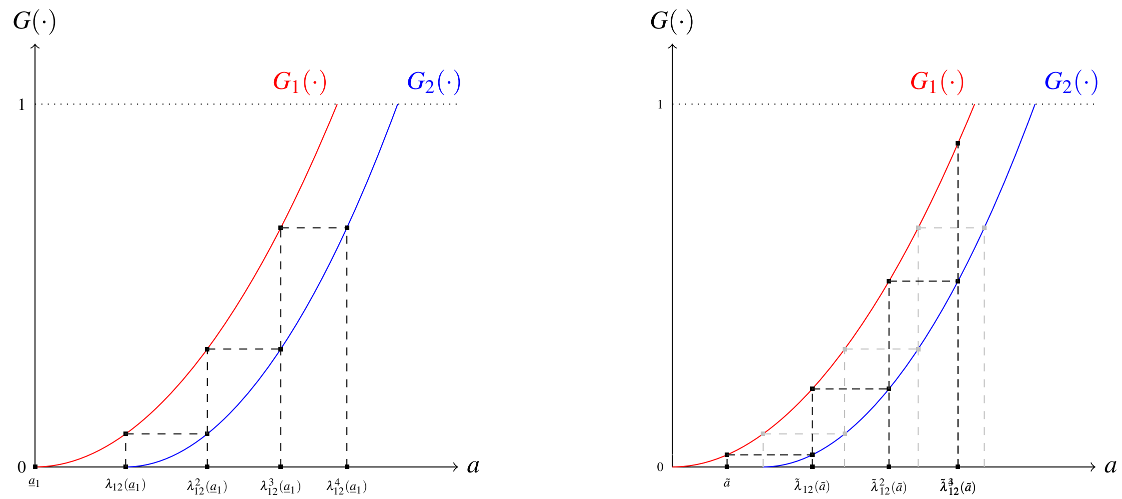
$$\mathscr{O}_{\underline{a}_1} = \left\{\underline{a}_1, \lambda_{12}(\underline{a}_1), \lambda_{12}^2(\underline{a}_1), \lambda_{12}^3(\underline{a}_1), \lambda_{12}^4(\underline{a}_1)\right\},$$

The function values of z' evaluated on $\mathscr{O}_{\underline{a}_1}$ are point identified following Lemma 1 (or Lemma 5):

$$\mathscr{B}_{z^o} = \left\{z^o, \gamma_{12}(z^o), \gamma_{12}^2(z^o), \gamma_{12}^3(z^o), \gamma_{12}^4(z^o)\right\},$$

where $\mathscr{B}_{z^o}$ is a new orbit defined as: $\mathscr{B}_{z^o} \equiv \{\gamma(z^o) | \gamma \in \Gamma_{z^o}\}$.

Figure 3: Partial identification if G_1 and G_2 do not cross.



(a) Partial identification: identified values.

(b) Partial identification: how to derive bounds.

Suppose our target is to bound $z'(\tilde{a})$, where $\tilde{a} \in \left(\underline{a}_1, \lambda_{12}(\underline{a}_1)\right)$. In order to characterize the bounds on $z'(\tilde{a})$, we first construct the preliminary bounds following Proposition 3 via the point-identified orbit $\mathscr{O}_{\underline{a}_1}$ (and $\mathscr{B}_{z^o}$), then propose a second-step iterative algorithm that refines the first-step preliminary bounds and converges to the fixed points stated in Lemma 6, based on which we derive the final pointwise sharp bounds following Theorem 2.

Starting from $\tilde{a}$, the orbit $\mathscr{O}_{\tilde{a}}$ also has a finite number of elements and is a monotone sequence, as shown in Figure 3b and expressed as:

$$\mathscr{O}_{\tilde{a}} = \left\{\tilde{a}, \tilde{\lambda}_{12}(\tilde{a}), \tilde{\lambda}_{12}^2(\tilde{a}), \tilde{\lambda}_{12}^3(\tilde{a}), \tilde{\lambda}_{12}^4(\tilde{a})\right\}.$$

Each element in this orbit satisfies:

$$\begin{aligned} &\tilde{\lambda}_{12}^l(\tilde{a}) \in \left(\lambda_{12}^l(\underline{a}_1), \lambda_{12}^{l+1}(\underline{a}_1)\right) \ \forall l \in \{0,1,2,3\}, \text{ and} \\ &\tilde{\lambda}_{12}^4(\tilde{a}) > \lambda_{12}^4(\underline{a}_1). \end{aligned} \tag{4.1}$$

Therefore, the function values of z' on $\mathscr{O}_{\tilde{a}}$ can be expressed as:

$$\mathscr{B}_{z'(\tilde{a})} = \left\{ z'(\tilde{a}), \tilde{\gamma}_{12}(z'(\tilde{a})), \tilde{\gamma}_{12}^{2}(z'(\tilde{a})), \tilde{\gamma}_{12}^{3}(z'(\tilde{a})), \tilde{\gamma}_{12}^{4}(z'(\tilde{a})) \right\}.$$

We want to note here that the generations of $\mathscr{O}_{\tilde{a}}$ and $\mathscr{B}_{z'(\tilde{a})}$ rely on $\tilde{\lambda}_{12}$ and $\tilde{\gamma}_{12}$ that are different from those to generate $\mathscr{O}_{\underline{a}_1}$ and $\mathscr{B}_{z^o}$, i.e., λ_{12} and γ_{12}, because they depend on different sequences of as: one on $\mathscr{O}_{\underline{a}_1}$ and the other on $\mathscr{O}_{\tilde{a}}$.

We can derive the upper and lower bounds for each element in $\mathscr{B}_{z'(\tilde{a})}$ due to the shape restriction on z' imposed by Assumption 2-(iii):

$$\begin{aligned} &\tilde{\gamma}_{12}^{l}(z'(\tilde{a})) \in \left[\gamma_{12}^{l+1}(z^o), \gamma_{12}^{l}(z^o)\right] \; \forall l \in \{0,1,2,3\}, \text{ and} \\ &\tilde{\gamma}_{12}^{4}(z'(\tilde{a})) \leq \gamma_{12}^{4}(z^o). \end{aligned}$$

Further, note that each element in $\mathscr{B}_{z'(\tilde{a})}$ can be mapped back to $z'(\tilde{a})$ as $z'(\tilde{a}) = \left(\tilde{\gamma}_{12}^{l}\right)^{-1}\left(\tilde{\gamma}_{12}^{l}(z'(\tilde{a}))\right)$. Since γ_{12} is strictly increasing under Assumption 2-(iv), we can convert the bounds on each element in $\mathscr{B}_{z'(\tilde{a})}$ to those on $z'(\tilde{a})$:

$$\begin{aligned} &z'(\tilde{a}) \in \left[\left(\tilde{\gamma}_{12}^{l}\right)^{-1}\left(\gamma_{12}^{l+1}(z^o)\right), \left(\tilde{\gamma}_{12}^{l}\right)^{-1}\left(\gamma_{12}^{l}(z^o)\right)\right] \; \forall l \in \{0,1,2,3\}, \text{ and} \\ &z'(\tilde{a}) \leq \left(\tilde{\gamma}_{12}^{4}\right)^{-1}\left(\gamma_{12}^{4}(z^o)\right). \end{aligned}$$

And we can derive the first-step lower and upper bounds of $z'(\tilde{a})$ as:

$$L_0(z'(\tilde{a}); z^o) \equiv \max\left\{\gamma_{12}(z^o), \left(\tilde{\gamma}_{12}\right)^{-1}\left(\gamma_{12}^{2}(z^o)\right), \left(\tilde{\gamma}_{12}^{2}\right)^{-1}\left(\gamma_{12}^{3}(z^o)\right), \left(\tilde{\gamma}_{12}^{3}\right)^{-1}\left(\gamma_{12}^{4}(z^o)\right)\right\}; \tag{4.2}$$

$$U_0(z'(\tilde{a}); z^o) \equiv \min\left\{z^o, \left(\tilde{\gamma}_{12}\right)^{-1}\left(\gamma_{12}(z^o)\right), \left(\tilde{\gamma}_{12}^{2}\right)^{-1}\left(\gamma_{12}^{2}(z^o)\right), \left(\tilde{\gamma}_{12}^{3}\right)^{-1}\left(\gamma_{12}^{3}(z^o)\right), \left(\tilde{\gamma}_{12}^{4}\right)^{-1}\left(\gamma_{12}^{4}(z^o)\right)\right\}, \tag{4.3}$$

where the subscript 0 indicates this is the first step.

The bounds in (4.2) and (4.3) can be further refined by the following iterative algorithm (where L_j and U_j represent the constructed bounds at the j-th iteration):

Step 1: For each $\hat{a} \in \left(\underline{a}_1, \lambda_{12}(\underline{a}_1)\right)$ but $\hat{a} \neq \tilde{a}$, we derive the first-step lower and upper bounds for $z'(\hat{a})$: $L_0(z'(\hat{a}); z^o)$ and $U_0(z'(\hat{a}); z^o)$.

Step 2: Using this $\hat{a}$ in the same way as $\underline{a}_1$, we construct a new pair of lower and upper bounds for $z'(\tilde{a})$: $L(z'(\tilde{a}); z'(\hat{a}))$ and $U(z'(\tilde{a}); z'(\hat{a}))$, which are used in each iteration.

Step 3: In the j-th iteration where $j \geq 1$, the updated lower and upper bounds on $z'(\tilde{a})$ are as follows:

$$L_j(z'(\tilde{a}); z^o) \equiv \max\left\{L_{j-1}(z'(\tilde{a}); z^o), \sup_{\substack{\hat{a} \in \left(\underline{a}_1, \lambda_{12}(\underline{a}_1)\right) \\ \hat{a} \neq \tilde{a}}} \left\{L\left(z'(\tilde{a}); L_{j-1}(z'(\hat{a}); z^o)\right)\right\}\right\};$$

$$U_j\left(z'\left(\tilde{a}\right);z^o\right)\equiv\min\left\{U_{j-1}\left(z'\left(\tilde{a}\right);z^o\right),\inf_{\substack{\hat{a}\in\left(\underline{a}_1,\lambda_{12}\left(\underline{a}_1\right)\right)\\ \hat{a}\neq\tilde{a}}}\left\{U\left(z'\left(\tilde{a}\right);U_{j-1}\left(z'\left(\hat{a}\right);z^o\right)\right)\right\}\right\}.$$

Step 4: By a symmetric argument, we also update the bounds on $z'\left(\hat{a}\right)$:

$$L_j\left(z'\left(\hat{a}\right);z^o\right)\equiv\max\left\{L_{j-1}\left(z'\left(\hat{a}\right);z^o\right),\sup_{\substack{\tilde{a}\in\left(\underline{a}_1,\lambda_{12}\left(\underline{a}_1\right)\right)\\ \tilde{a}\neq\hat{a}}}\left\{L\left(z'\left(\hat{a}\right);L_{j-1}\left(z'\left(\tilde{a}\right);z^o\right)\right)\right\}\right\};$$

$$U_j\left(z'\left(\hat{a}\right);z^o\right)\equiv\min\left\{U_{j-1}\left(z'\left(\hat{a}\right);z^o\right),\inf_{\substack{\tilde{a}\in\left(\underline{a}_1,\lambda_{12}\left(\underline{a}_1\right)\right)\\ \tilde{a}\neq\hat{a}}}\left\{U\left(z'\left(\hat{a}\right);U_{j-1}\left(z'\left(\tilde{a}\right);z^o\right)\right)\right\}\right\}.$$

Step 5: We repeat Steps 3-4 until all the bounds converge.

Step 6: If the iterative algorithm converges at multiple different points, we choose the tightest pair as the final lower and upper bounds for $z'\left(\tilde{a}\right)$.

Note that for other target $\tilde{a}$ where $\tilde{a}\in\left(\lambda_{12}^l\left(\underline{a}_1\right),\lambda_{12}^{l+1}\left(\underline{a}_1\right)\right)$ for $l=1,2,3$ or $\tilde{a}>\lambda_{12}^4\left(\underline{a}_1\right)$, we can use the same approach to bound $z'(\tilde{a})$ by noting that the orbit $\mathscr{O}_{\tilde{a}}$ has the same feature as (4.1).[25]

Lastly, the point and partial identification results on z' lead to those on F following Theorem 2-(ii).

4.2 Numerical Exercise

In this subsection, we provide a numerical exercise of our point and partial identification results. When $K=2$, the Tullock contest model in Example 4 have both crossing and non-crossing patterns, as discussed and numerically shown in Wasser (2013). Therefore, we choose the Tullock model for illustration. According to Wasser (2013) and Ewerhart (2014), the interior equilibrium is reduced to

$$t_i\cdot\mathbb{E}_{\mathbf{a}_{-i}}\left[\frac{\sum_{j\neq i}a_j}{\left(a_i+\sum_{j\neq i}a_j\right)^2}\right]=c'(a_i). \tag{4.4}$$

Denote $p=1/t$, which can be interpreted as the private cost, whose distribution is specified in this numerical illustration following Wasser (2013).

Given the complexity of the iterative algorithm presented in Section 4.1 and the potential cumulative numerical errors that may arise during the construction of bounds, we only present the preliminary bounds obtained in the first step for the purpose of illustrating partial identification (Proposition 3).

25 This partial identification approach to derive the preliminary bounds is also applied in D'Haultfœuille and Février (2020), however, our bounds are more involved due to the generality of the class of Bayesian games we consider and the iterative algorithm we propose. In contrast, their derived bounds are constants for all targets between two point identified values. Additionally, we use $K=2$ as an illustration and the results for $K\geq 3$ can be found in Section 3.3, while they only focus on $K=2$. Lastly, the partial identification we derive for F pool all bounds from different N_j together, while they do not, thus conjecturally their bounds for F could be wider.

It is important to emphasize that this numerical exercise underscores the strong dependence of our identification approach on the exclusion restriction specified in Assumption 3, as elaborated below.

4.2.1 When the Exclusion Restriction Holds

We first consider the scenario when the exclusion restriction holds. p is assumed to follow a uniform distribution on $[0.5, 2.5]$. The true cost function is quadratic: $c(x) = x^2/2$, leading to an identity cost derivative function: $c'(x) = x$. We approximate the equilibrium effort strategy $\nu(p) \equiv s(t)$ numerically by a discrete function on a grid of points in $[0.5, 2.5]$. The size of the grid is set to be $g = 5000$.[26] And the set of points is denoted as:

$$\hat{\mathbf{p}} = \{p^1, p^2, \cdots, p^g\}.$$

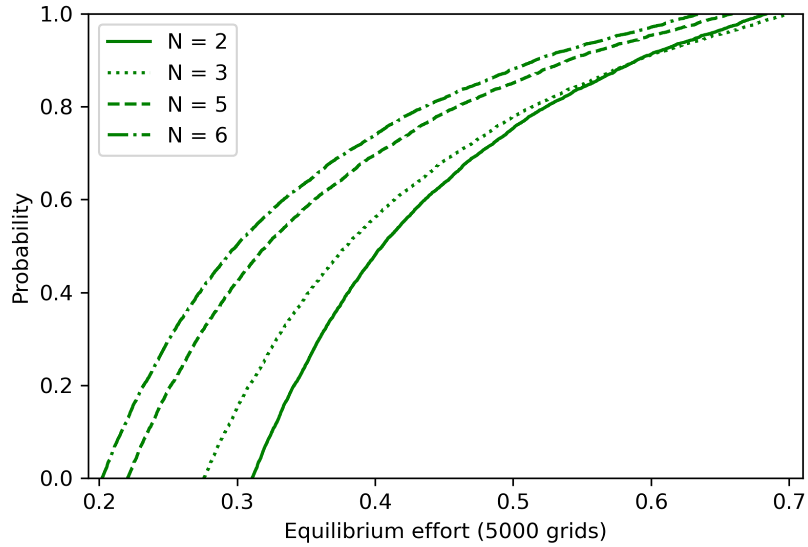
When the exclusion restriction holds, we consider the same $\hat{\mathbf{p}}$ for different numbers of contestants: $N = 2$, $N = 3$, $N = 5$, and $N = 6$. The numerical equilibrium efforts distributions are shown in Figure 4. When N increases from 2 to 3, the change corresponds to the one-crossing pattern; when N increases from 3 to 5 (or 6), the non-crossing pattern emerges. We apply the point identification approach to the crossing pattern, e.g., $N \in \{2, 3\}$ to identify the cost derivative function over the support of equilibrium efforts under $N = 3$. We also apply the partial identification result to two non-crossing patterns, e.g., $N \in \{3, 5\}$ and $N \in \{3, 6\}$, in order to identify the cost derivative function over the same support. In addition, we show identification results of the distribution of the private type t, denoted as F.

For point identification, we normalize the cost derivative at the intersection point to be the effort itself, since the true cost derivative function is an identity function represented by the 45-degree line. As shown in Figure 5a, the identified cost derivative function is more accurate as the effort value being evaluated is closer to the intersection point. The reason is that the point identification result relies on the action needed to link any a to the intersection point, as shown in Figure 2 where infinite composition(s) of the action λ_{ji} is necessary. And this can cause the numerical error induced by calculating the empirical CDF and the empirical quantile to accumulate. As the effort being identified gets further away from the intersection point, the cumulative error gets larger. The corresponding identified private type distribution is shown in Figure 6a, which shows similar patterns as the result of c': the numerical error accumulates as one moves away from the intersection point.

The partial identification results are shown in Figure 5b, where the upper and lower bounds of the cost derivative function are displayed. First of all, starting from the lower boundary of the support

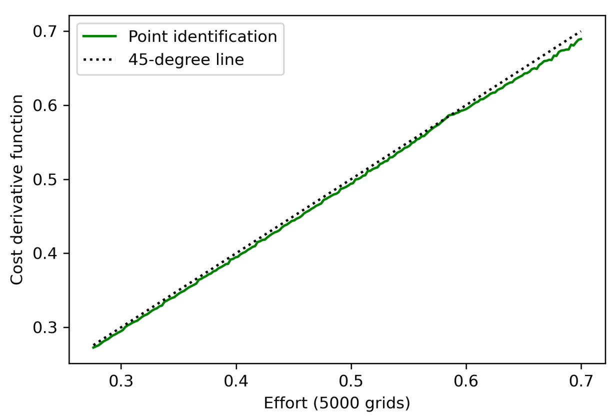
26 We also conduct the analysis when the number of grid points is $g = 500$ or $g = 1000$. It becomes evident that as the number of grid points increases, the accuracy of the point identification result improves. To save space, we do not report the results for $g = 500$ and $g = 1000$; however, these findings are available upon request.

Figure 4: When the exclusion restriction holds: effort distributions with different N

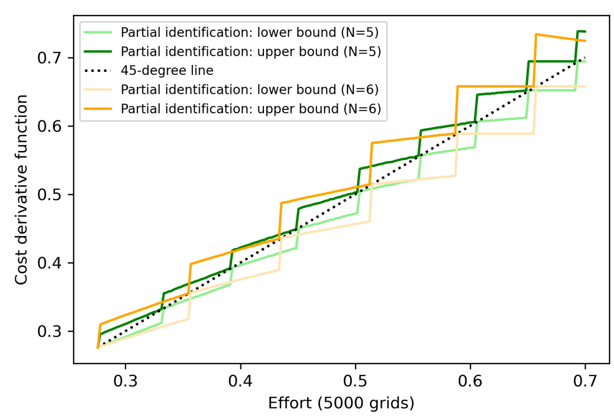


of equilibrium efforts under $N = 3$, there is a finite set of effort values evaluated at which the cost derivatives are point identified. The bounds are thus derived for each section between two point identified values. Compared to $N = 6$, the effort distribution with $N = 5$ is closer to that with $N = 3$. Therefore, there are more effort values at which the cost derivatives are point identified resulting in tighter bounds. The corresponding partial identification results of the private type distribution are shown in Figure 6b, which again shows similar patterns as those of c'.[27]

Figure 5: When the exclusion restriction holds: point and partial identification results of c'.



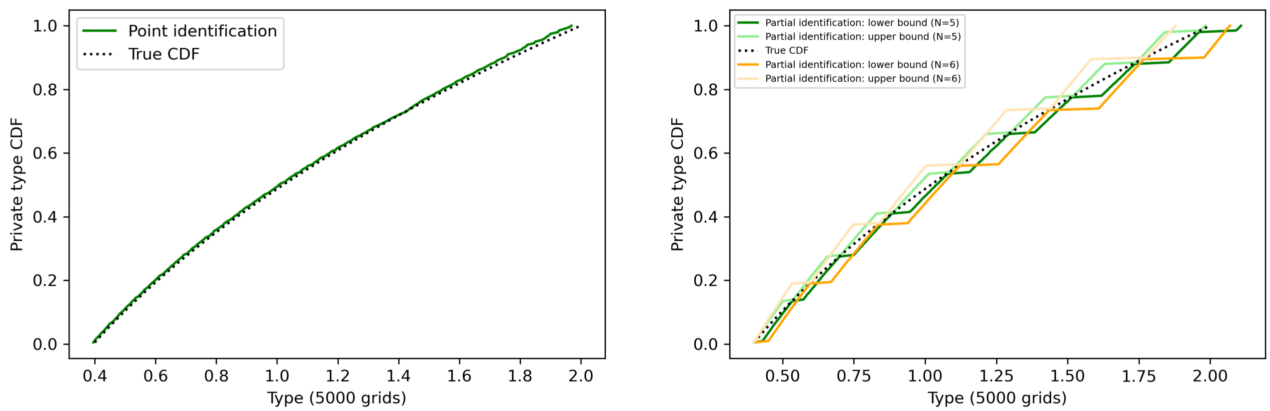
(a) Point identification of the cost derivative function.



(b) Partial identification of the cost derivative function.

27 Here, we do not pool the bounds derived from different sets of Ns together, as the identification is not exact, and the cumulative numerical errors make the bounds not compatible with each other for some regions of t.

Figure 6: When the exclusion restriction holds: point and partial identification results of F.



(a) Point identification of the private type distribution.

(b) Partial identification of the private type distribution.

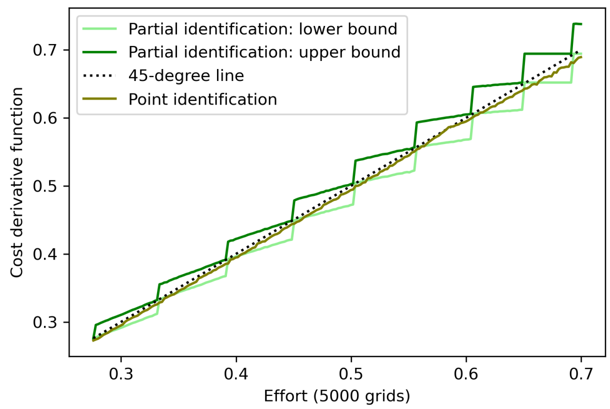
A few remarks regarding the results from the point identification and the partial identification approaches are in order. First, the point identification results are subject to the accumulation of numerical errors. In contrast, the bounds in the partial identification case are derived from the pairs of point identified values that are finite; hence the numerical error seems smaller relative to the case of point identification. Second, in the point identification case the normalized point is chosen to be the intersection point; while in the partial identification case, the normalized point is always the lower boundary of the support of equilibrium efforts under $N = 3$. Figure 7 shows the results from the point identification ($N \in \{2,3\}$) and the partial identification ($N \in \{3,5\}$). Roughly speaking, the two sets of results are compatible with each other, even though the point identification result is further away from the true function (45-degree line) as one moves away from the intersection point due to the cumulative numerical error. Nevertheless and importantly, in a local area around the intersection point, the point identification result is always inside the derived bounds, which holds true for both c' and F.

4.2.2 When the Exclusion Restriction Does Not Hold

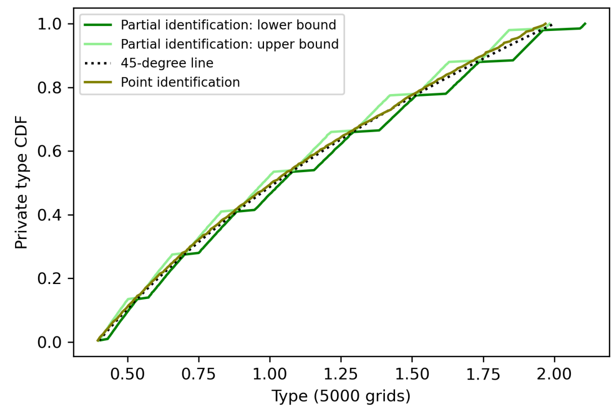
When the exclusion restriction does not hold, the private cost distribution varies as N increases. We assume that when $N = 2$, the cost follows a truncated standard normal distribution in $[0.5, 2.5]$; when $N = 3$, the cost follows a uniform distribution in $[0.5, 2.5]$; when $N = 5$, the cost follows a truncated exponential distribution in $[0.5, 2.5]$; when $N = 6$, the cost follows a truncated logistic distribution in $[0.5, 2.5]$.

The distributions of equilibrium efforts are shown in Figure 8. We apply the point identification approach to the one-crossing pattern, e.g., $N \in \{2,3\}$ to identify the cost derivative function and the

Figure 7: Comparison of the point and partial identification when the exclusion restriction holds.



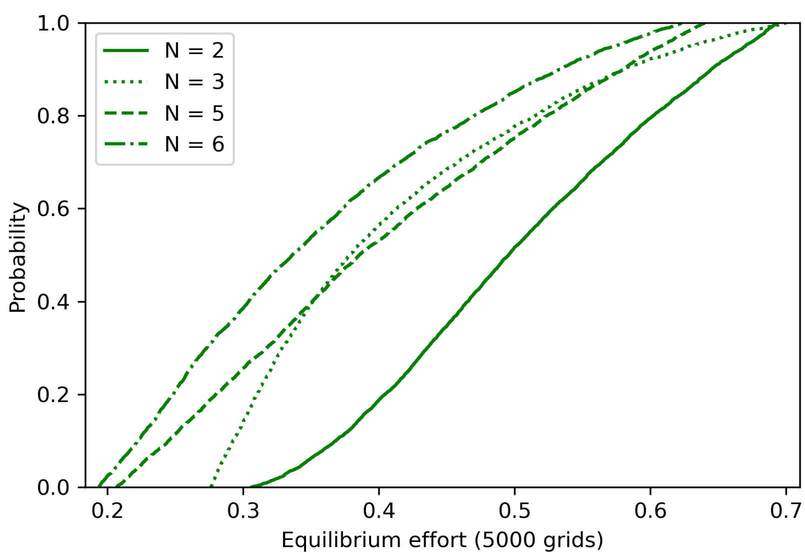
(a) The cost derivative function.



(b) The private type distribution.

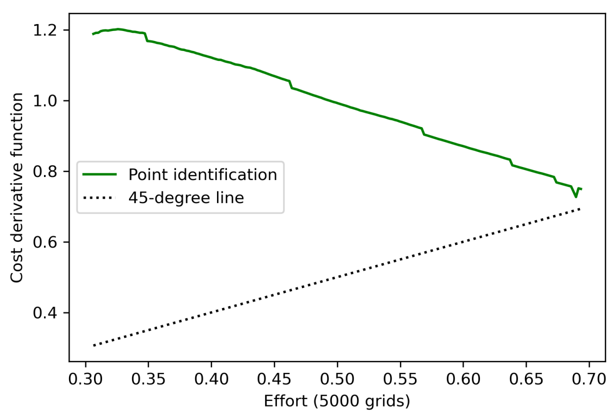
private type distribution over the support of equilibrium efforts and private types under $N = 2$ (Figures 9a and 10a). We also apply the partial identification result to two non-crossing patterns, e.g., $N \in \{2,5\}$, and $N \in \{2,6\}$, in order to identify the cost derivative function and the private type distribution under $N = 2$ (Figures 9b and 10b).

Figure 8: When the exclusion restriction does not hold: effort distributions with different N

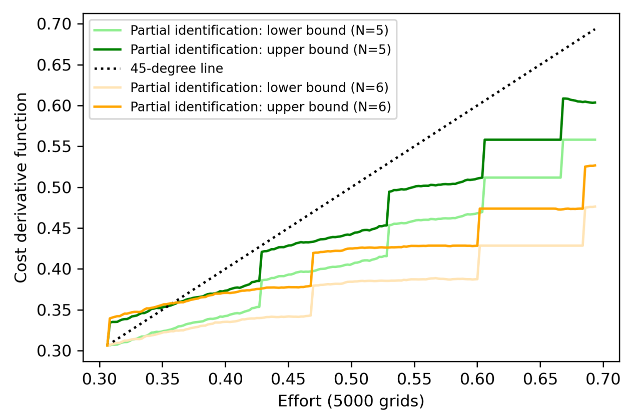


While the bounds from the partial identification shown in Figures 9b and 10b seem closer to the true function than the point identification result, the 45-degree line lies outside the identified sets. The point identification results shown in Figures 9a and 10a are substantially different from the true function: the point identified cost derivative function is not even increasing, and the point identified private type distribution is now a correspondence rather than a function with incorrect support. When the exclusion restriction does not hold, the bounds from different Ns are not compatible with each other. The derived bounds from $N = 2$ and $N = 5$ are not tighter and do not lie within those from $N = 2$ and $N = 6$, which is in sharp contrast with the results in Figures 5b and 6b. As a result, this can be used as a basis to propose a test for the exclusion restriction, provided that nonparametric set inference methods are developed.

Figure 9: When the exclusion restriction does not hold: point and partial identification results of c'.

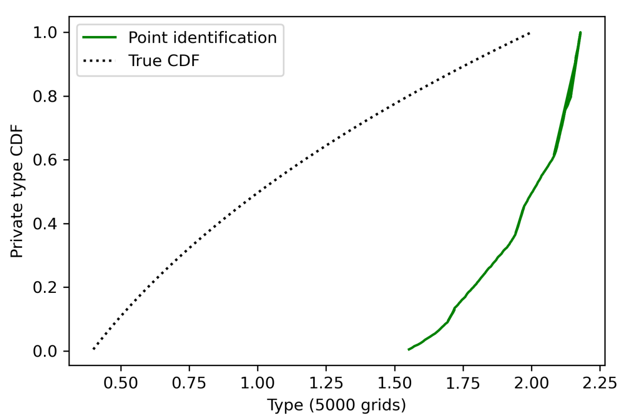


(a) Point identification of the cost derivative function.

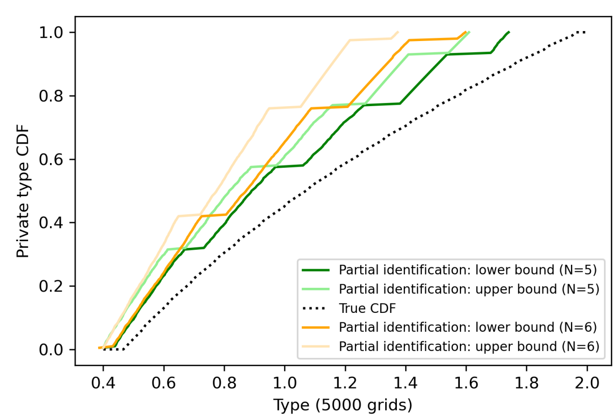


(b) Partial identification of the cost derivative function.

Figure 10: When the exclusion restriction does not hold: point and partial identification results of F.



(a) Point identification of the private type distribution.



(b) Partial identification of the private type distribution.

5 Extensions

5.1 Corner Solutions

In the previous sections, we only consider the interior solution case, where the first order condition yields Equation (3.3). However, in some empirical applications, *corner solutions* may occur. In this subsection, we use the contest model in Example 4 to discuss the nonparametric identification of the unknown structure $[F, c']$, given the exclusion restriction, when the contestant's equilibrium effort is allowed to be zero.[28] The first-order condition is now characterized by the Krush-Kuhn-Tucker (KKT) conditions:

$$t_i \cdot \mathbb{E}_{\mathbf{a}_{-i}} \left[\frac{\sum_{j \neq i} a_j}{(a_i + \sum_{j \neq i} a_j)^2} \right] = c'(a_i), \; if \; a_i > 0, \tag{5.1}$$

$$t_i \cdot \mathbb{E}_{\mathbf{a}_{-i}} \left[\frac{\sum_{j \neq i} a_j}{(a_i + \sum_{j \neq i} a_j)^2} \right] \leq c'(a_i), \; if \; a_i = 0. \tag{5.2}$$

Therefore, there exist both "inactive" contestants who make zero effort and obtain zero expected payoffs, as well as "active" ones whose efforts are strictly positive. Ewerhart (2014) shows that there exists a unique, symmetric, and monotone equilibrium strategy $s(\cdot)$ in the above Tullock contest, and at least one contestant remains active in equilibrium. For active contestants, $s(\cdot)$ is strictly increasing since the strict supermodularity condition (Assumption 2-(iv)) is satisfied.

To apply the nonparametric identification approaches proposed in Section 3, we can proceed in the same way as before, by looking at how the effort distributions behave for the private types who are active when N varies exogenously.

It is worth noting that in the corner solution case, $c'(\cdot)$ and $F(\cdot)$ can only be point or partially identified over a restricted support, due to the reason that the identification approaches can only be adopted for strictly positive efforts in Equation (5.1). For instance, in the Tullock contest, let τ_j^* be the probability corresponding to the upper boundary of zero efforts in the contest with N_j contestants (also the lower boundary of non-zero efforts in this contest). Then $c'(\cdot)$ is only identified over the restricted support $\cup_{j=1}^{K} \left[a_j(\tau_j^*), \bar{a}_j \right]$, and $F(\cdot)$ is identified over the restricted support $\left[\min_{j \in \{1, \cdots, K\}} \left\{ t(\tau_j^*) \right\}, \bar{t} \right]$.

5.2 Asymmetric Private Type Distribution

We now consider the asymmetric case where each player i has her private type drawn from different $F_i(\cdot)$ over the support $[\underline{t}_i, \bar{t}_i]$ with $F_i(\cdot) \in \mathscr{F}$ in Assumption 1. We maintain the exclusion restriction as in Assumption 3. The strictly MPSNE of player i is denoted by $s_i(t_i) = a_i$, with equilibrium action CDF $G_i(\cdot|N)$ over the support $[\underline{a}_i(N), \bar{a}_i(N)]$, with $G_i(\cdot|N) \in \mathscr{G}$. The first order condition (an adaption

28 Here we focus on the identification of $c'(\cdot)$, the derivative of the cost function.

of Equation (3.3)) now becomes:

$$t_i(\tau)\cdot \mathbf{E}_{\mathbf{a}_{-N}}\left[\frac{\partial x(a_i(\tau),\mathbf{a}_{-N})}{\partial a_i}\middle| N\right]+\mathbf{E}_{\mathbf{a}_{-N}}\left[\frac{\partial y(a_i(\tau),\mathbf{a}_{-N})}{\partial a_i}\middle| N\right]+z'\left[a_i(\tau)\right]=0, \tag{5.3}$$

where the bolded $\mathbf{E}_{\mathbf{a}_{-N}}$ denotes the expectation over the joint action distribution except player i: $\mathbf{G}_{-i}(\cdot,\cdots,\cdot|N)=\Pi_{j\neq i}G_j(\cdot|N)$. We consider the interior solution case.

Our identification problem is to recover the structure $[F_1(\cdot),\cdots,F_{N_j}(\cdot);z'(\cdot)]$, given the equilibrium action vector $\left\{a_{j,1},\cdots,a_{j,N_j}\right\}$, and action CDFs $\left\{G_{j,1}(\cdot),\cdots,G_{j,N_j}(\cdot)\right\}$ for $j\in\{1\cdots,K\}$.

Under Assumption 3, it is crucial to cancel out $t_i(\tau)$ in Equation (5.3) above, as in the beginning of Section 3.1. Therefore, one necessary constraint to apply the identification approach is to assume that there are N_0 common players with $1\leq N_0\leq \min_{j\in\{1,\cdots,K\}}\left\{N_j\right\}$. We assume that these common players are indexed as the first N_0 players. The point/partial identification procedure is thus as follows:

Step 1: Consider any pair $(N_1,N_2)\in\mathcal{N}^2$. Choose an arbitrary common player i, with private type distribution $F_i(\cdot)$. By canceling out $t_i(\tau)$, we get the following equation

$$\begin{aligned}&\mathbf{E}_{\mathbf{a}_{-N_1}}\left[\frac{\partial y(a_{1,i}(\tau),\mathbf{a}_{-N_1})}{\partial a_{1,i}}\middle|N_1\right]\mathbf{E}_{\mathbf{a}_{-N_2}}\left[\frac{\partial x(a_{2,i}(\tau),\mathbf{a}_{-N_2})}{\partial a_{2,i}}\middle|N_2\right]-\mathbf{E}_{\mathbf{a}_{-N_2}}\left[\frac{\partial y(a_{2,i}(\tau),\mathbf{a}_{-N_2})}{\partial a_{2,i}}\middle|N_2\right]\mathbf{E}_{\mathbf{a}_{-N_1}}\left[\frac{\partial x(a_{1,i}(\tau),\mathbf{a}_{-N_1})}{\partial a_{1,i}}\middle|N_1\right]\\&=z'\left[a_{2,i}(\tau)\right]\mathbf{E}_{\mathbf{a}_{-N_1}}\left[\frac{\partial x(a_{1,i}(\tau),\mathbf{a}_{-N_1})}{\partial a_{1,i}}\middle|N_1\right]-z'\left[a_{1,i}(\tau)\right]\mathbf{E}_{\mathbf{a}_{-N_2}}\left[\frac{\partial x(a_{2,i}(\tau),\mathbf{a}_{-N_2})}{\partial a_{2,i}}\middle|N_2\right].\end{aligned}$$

Step 2: Applying the corresponding identification approaches in Section 3, $z'(\cdot)$ is nonparametrically identified over $\cup_{j\in\{1,\cdots,K\}}\left[\underline{a}_{j,i},\bar{a}_{j,i}\right]$, and $F_i(\cdot)$ is nonparametrically identified over $[\underline{t}_i,\bar{t}_i]$, with $F_i(\cdot)=G_{j,i}\left(\xi_{j,i}^{-1}(\cdot)\right)$ for $j\in\{1,\cdots,K\}$.

Step 3: Varying the index i within the group of common players, $z'(\cdot)$ is identified over $\cup_{i\in\{1,\cdots,N_0\}}\cup_{j\in\{1,\cdots,K\}}\left[\underline{a}_{j,i},\bar{a}_{j,i}\right]$.[29]

Step 4: Choosing an arbitrary remaining contestant r, with private type distribution $F_r(\cdot)$. Thus when $a_{j,r}\in\cup_{i\in\{1,\cdots,N_0\}}\cup_{j\in\{1,\cdots,K\}}\left[\underline{a}_{j,i},\bar{a}_{j,i}\right]$, the corresponding private type t_r can be recovered. As a result, $F_r(\cdot)$ is identified over such values of t_r.

5.3 Asymmetric Function z

We now consider the asymmetric case where N players draw their private types from a common CDF $F(\cdot)$ over $[\underline{t},\bar{t}]$, but have different functions $z_i(\cdot)$ satisfying Assumption 2. Although the latent private type distribution is common, each player i can have a different equilibrium strategy $s_i(t_i)=a_i$ with different equilibrium action CDF $G_i(\cdot|I)$ over $[\underline{a}_i(N),\bar{a}_i(N)]$.

29 If the support $\cup_{j\in\{1,\cdots,K\}}\left[\underline{a}_{j,i},\bar{a}_{j,i}\right]$ of different common player i overlaps, varying the index i may potentially tighten the bounds of $z'(\cdot)$ in the case of partial identification.

Suppose that there are N_0 common players where $1 \leq N_0 \leq \min_{j\in\{1,\cdots,K\}} \{N_j\}$. We assume that these common players are indexed as the first N_0 players. The point/partial identification approach is as follows:

Step 1: Consider any pair $(N_1, N_2) \in \mathcal{N}^2$. Choose an arbitrary common player i, with the function $z_i(\cdot)$. By canceling out $t(\tau)$, we get the following equation

$$\mathbf{E}_{\mathbf{a}_{-N_1}}\left[\frac{\partial y(a_{1,i}(\tau),\mathbf{a}_{-N_1})}{\partial a_{1,i}}\bigg|N_1\right]\mathbf{E}_{\mathbf{a}_{-N_2}}\left[\frac{\partial x(a_{2,i}(\tau),\mathbf{a}_{-N_2})}{\partial a_{2,i}}\bigg|N_2\right]-\mathbf{E}_{\mathbf{a}_{-N_2}}\left[\frac{\partial y(a_{2,i}(\tau),\mathbf{a}_{-N_2})}{\partial a_{2,i}}\bigg|N_2\right]\mathbf{E}_{\mathbf{a}_{-N_1}}\left[\frac{\partial x(a_{1,i}(\tau),\mathbf{a}_{-N_1})}{\partial a_{1,i}}\bigg|N_1\right]$$
$$=z_i'[a_{2,i}(\tau)]\,\mathbf{E}_{\mathbf{a}_{-N_1}}\left[\frac{\partial x(a_{1,i}(\tau),\mathbf{a}_{-N_1})}{\partial a_{1,i}}\bigg|N_1\right]-z_i'[a_{1,i}(\tau)]\,\mathbf{E}_{\mathbf{a}_{-N_2}}\left[\frac{\partial x(a_{2,i}(\tau),\mathbf{a}_{-N_2})}{\partial a_{2,i}}\bigg|N_2\right].$$

Step 2: Applying the identification approach in Section 3, $z_i'(\cdot)$ is nonparametrically identified over $\cup_{j\in\{1,\cdots,K\}}\left[\underline{a}_{j,i}, \bar{a}_{j,i}\right]$, and $F(\cdot)$ is nonparametrically identified over $[\underline{t}, \bar{t}]$, with $F(\cdot) = G_{j,i}\left(\xi_{j,i}^{-1}(\cdot)\right)$ for $j \in \{1, \cdots, K\}$.[30]

Step 3: Choose an arbitrary remaining contestant r. Thus from player r's first order condition, $z_r'(\cdot)$ is identified over $\cup_{j\in\{1,\cdots,K\}}\left[\underline{a}_{j,r}, \bar{a}_{j,r}\right]$.

5.4 Asymmetric Private Type Distribution and Function z

We now consider the asymmetric Bayesian game, where player i has own private type CDF $F_i(\cdot)$ over $[\underline{t}_i, \bar{t}_i]$ and own function $z_i(\cdot)$ satisfying Assumptions 1 and 2, respectively. Each player i has a different equilibrium action strategy $s_i(t_i) = a_i$ with a different action distribution $G_i(\cdot|N)$ over $[\underline{a}_i(N), \bar{a}_i(N)]$.

Suppose that there are N_0 common players where $1 \leq N_0 \leq \min_{j\in\{1,\cdots,K\}} \{N_j\}$. In this setup, only the private type CDFs $\{F_1(\cdot), \cdots, F_{N_0}(\cdot)\}$, and functions $\{z_1(\cdot), \cdots, z_{N_0}(\cdot)\}$ of common players can be nonparametrically and point/partially identified over their corresponding identified supports.

5.5 Asymmetric Function x

We now consider the extension to allow for each player i to have different $x_i(a_i, \mathbf{a}_{-i})$.[31] It is important to note that this function has to vary with N, which implies that for player i, when the number of participants change, x_i has to change as well, in order to cause the variation of the expectation expressed as a multiple integral.

Since x_i is assumed to be known to the researchers, its asymmetry does not add any more complication to the identification problem as it does not influence the cancellation of private type for common players when the first-order conditions for common players are stacked, which is the key insight we

30 Varying the index i may potentially tighten the bounds of $F(\cdot)$ in the case of partial identification.

31 The same argument applies to when each player has own $y_i(a_i, \mathbf{a}_{-i})$, thus is omitted here.

use in our identification strategy. Specifically, when asymmetric x_i appears together with asymmetric private type CDF F_i, the procedure in Section 5.2 can be adopted. Moreover, when both x_i and z_i are asymmetric, we can utilize the procedure in Section 5.3. Lastly, when the functions x_i, F_i and z_i are all asymmetric, the procedure in Section 5.4 can be applied.

Furthermore, as mentioned in Section 2.1, the deterministic and known function of t_i appearing in payoff functions in all three forms, i.e. $m_i(t_i)$, can be relaxed to be asymmetric, for the same reason as in the asymmetric x_i case. As a result, asymmetric m_i can be combined with asymmetric F_i or asymmetric z_i (along with asymmetric x_i), which will not affect the identification results in this section.

5.6 Unobserved Heterogeneity

In this subsection, we discuss the extension of the benchmark model to the case with unobserved game heterogeneity, which is modeled as a random variable U drawn independently from the CDF $F_U(\cdot)$ with the support denoted by $\mathscr{U}$. The realization of the unobserved heterogeneity u is common knowledge among players in the Bayesian game, but as econometricians we do not observe this u. Thus conditional on this u, players act as in the setup of our benchmark model, since their private types are conditionally independent. As a consequence, the properties of strictly MPSNE follow Proposition 1, given the conditional private type distribution $F(\cdot|u) \in \mathscr{F}$ defined in Assumption 1. Therefore, conditional on this u, the distribution of equilibrium actions denoted as $G(\cdot|u,N)$ satisfies the properties in Definition 1, i.e. $G(\cdot|u,N) \in \mathscr{G}$ for each $N \in \mathscr{N}$. Thus, the identification problem now becomes how to identify the unknown structure conditional on the realization $U = u$, together with the unknown distribution of the unobserved heterogeneity, i.e. $F_U(\cdot)$. Note that now the unknown private type's CDF is conditional on this u, thus $F(\cdot|u)$, while the other unknown function $z(\cdot)$ is unrelated to u.

In order to discuss the identification problem, we impose the following assumptions, which hold throughout this section.

Assumption 4

(i) Conditional exclusion restriction: for all $N \in \mathscr{N}$, $F(\cdot|u,N) = F(\cdot|u)$.

(ii) Stochastic monotonicity restriction: conditional on u, the private type is strictly increasing in u with respect to the first order.

There are two challenges in nonparametric identification of the model primitives with the unobserved heterogeneity. First, we need to identify the distribution of actions conditional on the unobserved heterogeneity and the distribution of the unobserved heterogeneity, given the unconditional distribution of actions identified in the data, for each N. Second, after identifying the above distributions, we have to know how to match the conditional distributions of actions for N_j, denoted as $G_j(\cdot|u)$, on the realization of the unobserved heterogeneity, or when $G_j(\cdot|u)$ corresponds to the same realization u for $j \in \{1, \cdots\}$, in order to apply the identification approaches described in Section 3.

Assumption 4-(ii) is needed for both discrete and continuous u in order to exploit support variation, see Hu, McAdams, and Shum (2013), Gentry and Li (2014), and Grundl and Zhu (2019) in the auction context. Particularly, the strict inequality is assumed to hold at the upper boundary of t, i.e., $\bar{t}(u)$ is strictly increasing in u. In general, that the private type is strictly FOSD-increasing with respect to the unobserved heterogeneity u does not necessarily imply that the equilibrium action is also FOSD-increasing in u.[32] If the Bayesian game under consideration is a game with strategic substitutes, or the payoff function satisfies

$$\frac{\partial^2 \pi(a_i, \mathbf{a}_{-i}, t_i)}{\partial a_i \partial a_j} = \frac{\partial^2 x(a_i, \mathbf{a}_{-i})}{\partial a_i \partial a_j} + \frac{\partial^2 y(a_i, \mathbf{a}_{-i})}{\partial a_i \partial a_j} \leq 0, \textit{ almost everywhere}, \forall i \neq j,$$

this stochastic monotonicity restriction on the private type with respect to the unobserved heterogeneity implies that the equilibrium action at a fixed τ: $a(\tau; u) = s(t(\tau); u)$ as a function of u, is increasing in u. Specifically, at the upper boundary, or $\tau = 1$, $\bar{a}(u)$ is strictly increasing in u. After establishing the monotonicity of the equilibrium action with respect to the unobserved heterogeneity, we can proceed to discuss the identification problem; we need to distinguish between the cases of discrete and continuous u, in order to extend Grundl and Zhu (2019) who deals with first-price auctions with risk aversion and unobserved auction heterogeneity to our setting.

If u is discrete, with finite support, i.e., the support of u is $1, 2, \cdots, U$ with $U < \infty$, we need at least two N, thus, $K \geq 2$; for each one, two randomly selected actions are observed. For different us, the upper boundaries of observed actions are distinguishable. Thus, we can identify the U action distributions conditional on one value of u for each N. Furthermore, the action distribution for each N can be sorted by the upper boundaries of their supports. If two different numbers of players correspond to the same u, they should have the same rank. Therefore, they should have the same private type distributions conditional on this u. After matching the games on the value of u, we can apply the identification approaches in the previous section to identify the private type distributions. By doing this for each u, we can obtain the final identification.

If u is continuous, for each N we need three randomly selected actions. Besides, in order to use the results on identification of triangular non-separable models, we need to impose an additional

32 An example can be found in Aryal and Zincenko (2023), who consider a Cournot-oligopoly model with private information of firms and linear demand function faced by firms. They establish identification of the model primitives, including the distribution of the unobserved heterogeneity. However, since their Cournot model does not have an unknown function like the unknown cost function $c(a_i)$ in Example 1, our models are different and it is unclear whether the strict monotonicity of equilibrium strategies still holds in our case.

assumption that the lower boundary of the private type $\underline{t}(u)$ is also strictly increasing in u, together with a normalization condition such as $\underline{t}(u) = u$. With these extra conditions, we ensure that the lower and upper boundaries of equilibrium actions are both strictly increasing with u, for each N. As a result, we can adopt the support variation approach to identify the distribution of actions conditional on u, together with the distribution of U. Moreover, it guarantees that $G_j(\cdot|u)$ with different j is conditional on the same u if the support has the same lower boundary, which fits into the identification approach proposed in Section 3.

5.7 Endogenous Participation

Our identification strategy has relied on the exclusion restriction in the form of an exogenous players' participation (Assumption 3). However, this restriction is no longer valid if players' private types are influenced by the competition level of the game, i.e., the number of players. In this subsection, we consider the case when there is unobserved heterogeneity ε at the game level that affects players' participation decisions, leading to the endogenous participation.[33] Specifically, we assume $N = N(\mathbf{Z}, \varepsilon)$, where $\mathbf{Z}$ is a vector of observed characteristics of the game. We discuss two empirically related scenarios regarding the correlation between the observed $\mathbf{Z}$ and the unobserved ε.

The first scenario is when $t \perp\!\!\!\perp \varepsilon|\mathbf{Z}$, thus the unobserved heterogeneity ε only affects players' participation, but not their private types, because the distribution of private types $F(\cdot|\mathbf{Z}, \varepsilon) = F(\cdot|\mathbf{Z})$. Thus for any given $\mathbf{Z}$, we obtain the exclusion restriction $F(\cdot|N, \mathbf{Z}) = F(\cdot|\mathbf{Z})$, but in the equilibrium, the action distribution $G(\cdot|N, \mathbf{Z})$ still depends on N. As a consequence, we can exploit the variation from two numbers of players $N_1 < N_2$ as before, but note that now they have the same characteristics $\mathbf{Z}$. Our point and partial identification approaches can thus be adopted in a straightforward manner.

The second scenario is more involved, when the unobserved heterogeneity ε is correlated with the private type.[34] We assume the availability of instrumental variable W such that $N = N(\mathbf{Z}, W, \varepsilon)$, and $t \perp\!\!\!\perp W|(\mathbf{Z}, \varepsilon)$. As a result, the private type distribution satisfies $F(\cdot|\mathbf{Z}, W, \varepsilon) = F(\cdot|\mathbf{Z}, \varepsilon)$. Thus for any fixed $(\mathbf{Z}, \varepsilon)$, we want to utilize the exclusion restriction $F(\cdot|N, \mathbf{Z}, \varepsilon) = F(\cdot|\mathbf{Z}, \varepsilon)$, in order to use the

33 For Bayesian game models, both theoretical and empirical studies on players' participation have been limited. To the best of our knowledge, the theoretical papers (Fu, Jiao, and Lu (2015), Gu, Hehenkamp, and Leininger (2019), and Jia and Sun (2021)) that study players' entry in contest models consider the complete information framework, thus not a Bayesian game. On the empirical side, we only note a few exceptions including Kawai and Sunada (2015) who estimate a dynamic model to analyze the campaign finance of electoral candidates, together with the challenger's selective entry decision and Zhao (2020) who constructs a contest model with both incomplete information and endogenous entry to study the U.S. Senate elections. Therefore, our benchmark model has been tailored to the current state of the literature. Nevertheless, we extend our benchmark result to the case of endogenous participation here.

34 Note that this situation is different from the case of the unobserved heterogeneity that enters directly players' private type distribution in Section 5.6. Here we consider the scenario where the unobserved heterogeneity affects private types through players' participation.

identification approaches, because in equilibrium, the action distribution is $G(\cdot|N,\mathbf{Z},\varepsilon)$. We follow Guerre, Perrigne, and Vuong (2009) to impose the following two conditions:

(i) $\varepsilon = N - \mathbb{E}[N|\mathbf{Z},W]$.

(ii) $W = h(X,\varepsilon)$, *where* $h(\cdot,\cdot)$ *is strictly increasing in* ε, $X \perp\!\!\!\perp \varepsilon$, *and* X *cannot be a subset of* $\mathbf{Z}$.

Under either one of the above conditions, we can identify ε as a first step, in order to identify the other structure of the model. When condition (i) holds, N is known and $E[N|\mathbf{Z},W]$ is identified. Therefore, ε can be treated as the error term and is identifiable. When condition (ii) holds, the function $h(\cdot,\cdot)$ can be identified following Matzkin (2003), and ε can be identified as the inverse of the function, i.e., $\varepsilon = h^{-1}(X,W)$. After ε is recovered, for any given $(\mathbf{Z},\varepsilon)$, under the exclusion restriction $F(\cdot|N,\mathbf{Z},\varepsilon) = F(\cdot|\mathbf{Z},\varepsilon)$, our identification approaches are applicable by noting that $G(\cdot|N,\mathbf{Z},\varepsilon)$ depends on N.

6 Conclusions

In this paper, we have explored a general approach to studying nonparametric identification of general Bayesian games with continuous payoff functions, which include many interesting applications. We characterize conditions under which we can establish either point or partial identification of the model primitives, which are the payoff function and the private type distribution. Our identification results are positive. To be more precise, we show that under the exclusion restriction in the form of an exogenous players' participation, in general, point identification can be established when the nonfreeness property holds. Conversely, when this property is not met, partial identification is generally attainable, and pointwise sharp bounds are constructed. We also extend the identification results to accommodate corner solutions, asymmetric players, unobserved heterogeneity, and endogenous participation, thus making our results applicable to a broad class of empirically relevant Bayesian games. As such, we have presented positive identification results and a general econometric framework for the structural analysis of general Bayesian games.[35]

Since our identification results are constructive, they can provide a basis for nonparametric estimation and inference (including testing for the exclusion restriction as discussed in Section 4.2.2) in the class of Bayesian games we consider. In the point identification case when the nonfreeness property holds, the estimation method proposed in Zincenko (2018) in estimating the model primitives in the first-price auction model with risk averse bidders can be extended to our case using our

35 While we focus on the games with incomplete information, games with complete information can also be included in our framework. This can be seen from replacing the expected derivatives of function $x(\cdot,\cdot)$ appearing in the left hand side difference of the two first order conditions in the equality relation (3.6) by just the derivatives of function $x(\cdot,\cdot)$, without the expectation. This does not affect our identification approach; thus our identification result still holds.

point identification results. On the other hand, in the partial identification case when the nonfreeness property fails, the nonparametric bounds can be consistently estimated by replacing the bounds by the nonparametric estimators of the point identified quantities. However, it raises new and challenging questions about inference for nonparametrically partially identified models, as the recent advances in inference on incomplete or partially identified econometric models have mainly focused on the parametric or semiparametric framework (e.g. Chernozhukov, Hong, and Tamer (2007), Ciliberto and Tamer (2009), Beresteanu, Molchanov, and Molinari (2011), Galichon and Henry (2011), and Chesher and Rosen (2017), among others), thus will be left for future research.[36]

36 D'Haultfœuille and Février (2020) establish consistency of their nonparametric estimator of the bounds, and suggest to use bootstrap to make inference. They then use their nonparametric bound estimates to specify parametric forms for the underlying structural elements and conduct structural estimation using the parametric approach.

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