5.B.2 Suppose first that Y exhibits constant returns to scale. Let $z \in \mathbb{R}_+^{L-1}$ and $\alpha > 0$. Then $(-z, f(z)) \in Y$. By the constant returns to scale, $(-\alpha z, \alpha f(z)) \in Y$. Hence $\alpha f(z) \leq f(\alpha z)$. By applying this inequality to αz in place of z and α^{-1} in place of α, we obtain $\alpha^{-1} f(\alpha z) \leq f(\alpha^{-1}(\alpha z)) = f(z)$, or $f(\alpha z) \leq \alpha f(z)$. Hence $f(\alpha z) = \alpha f(z)$. The homogeneity of degree one is thus obtained.

Suppose conversely that $f(\cdot)$ is homogeneous of degree one. Let $(-z, q) \in Y$ and $\alpha \geq 0$, then $q \leq f(z)$ and hence $\alpha q \leq \alpha f(z) = f(\alpha z)$. Since $(-\alpha z, f(\alpha z)) \in Y$, we obtain $(-\alpha z, \alpha q) \in Y$. The constant returns to scale is thus obtained.

5.C.10

(a) $$c(w,q) = \begin{cases} qw_1 & \text{if } w_1 \leq w_2; \\ qw_2 & \text{if } w_1 > w_2. \end{cases}$$

$$z(w,q) = \begin{cases} (q,0) & \text{if } w_1 < w_2; \\ \{(z_1,z_2) \in \mathbb{R}_+^2 : z_1 + z_2 = q\} & \text{if } w_1 = w_2; \\ (0,q) & \text{if } w_1 > w_2. \end{cases}$$

(b) $c(w,q) = (w_1 + w_2)q$. $z(w,q) = (q,q)$.

(c) $c(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(1-1/\rho)}$.

$z(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(-1/\rho)}(w_1^{1/(\rho-1)}, w_2^{1/(\rho-1)})$.

5.C.13 Denote the production function of the firm by $f(\cdot)$, then its optimization problem is

$$\text{Max}_{(z_1,z_2)\geq 0}\ pf(z_1,z_2) \quad \text{s.t.} \quad w_1z_1 + w_2z_2 \leq C.$$

This is analogous to the utility maximization problem in Section 3.D and the function $R(\cdot)$ corresponds to the indirect utility function. Hence, analogously to Roy's identity (Proposition 3.G.4), the input demands are obtained as

$$-\frac{1}{\nabla_C R(p,w,C)} \nabla_w R(p,w,C) = (\alpha C/w_1,\ (1-\alpha)C/w_2).$$

6.B.1 Suppose first that $L \succ L'$. A first application of the independence axiom (in the "only-if" direction in Definition 6.B.4) yields

$$\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

If these two compound lotteries were indifferent, then a second application of the independence axiom (in the "if" direction) would yield $L' \succsim L$, which contradicts $L \succ L'$. We must thus have

$$\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''.$$

Suppose conversely that $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$, then, by the independence axiom, $L \succsim L'$. If these two simple lotteries were indifferent, then the independence axiom would imply

$$\alpha L' + (1-\alpha)L'' \succsim \alpha L + (1-\alpha)L'',$$

a contradiction. We must thus have $L \succ L'$.

Suppose next that $L \sim L'$, then $L \succsim L'$ and $L' \succsim L$. Hence by applying the independence axiom twice (in the "only if" direction), we obtain

$$\alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''.$$

Conversely, we can show that if $\alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''$, then $L \sim L'$.

For the last part of the exercise, suppose that $L \succ L'$ and $L'' \succ L'''$, then, by the independence axiom and the first assertion of this exercise,

$$\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$$

and

$$\alpha L' + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''.$$

Thus, by the transitivity of $\succ$ (Proposition 1.B.1(i)),

$$\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''.$$