3.D.3 (a) We shall prove that for every $p \in \mathbb{R}^L_{++}$, $w \geq 0$, $\alpha \geq 0$, and $x \in \mathbb{R}^L_+$, if

$x = x(p,w)$, then $\alpha x = x(p,\alpha w)$. Note first that $p\cdot(\alpha x) \leq \alpha w$, that is, αx is affordable at $(p,\alpha w)$. Let $y \in \mathbb{R}^L_+$ and $p\cdot y \leq \alpha w$. Then $p\cdot(\alpha^{-1}y) \leq w$. Hence $u(\alpha^{-1}y) \leq u(x)$. Thus, by the homogeneity, $u(y) \leq u(\alpha x)$. Hence $\alpha x = x(p,\alpha w)$.

By this result,

$$v(p,\alpha w) = u(x(p,\alpha w)) = u(\alpha x(p,w)) = \alpha u(x(p,w)) = \alpha v(p,w).$$

Thus the indirect utility function is homogeneous of degree one in w.

Given the above results, we can write $x(p,w) = wx(p,1) = w\tilde{x}(p)$ and $v(p,w) = wv(p,1) = w\tilde{v}(p)$. Exercise 2.E.4 showed that the wealth expansion path $\{x(p,w)\colon w > 0\}$ is a ray going through $\tilde{x}(p)$. The wealth elasticity of demand $\varepsilon_{\ell w}$ is equal to 1.

(b) We first prove that for every $p \in \mathbb{R}^L_{++}$, $w \geq 0$, and $\alpha \geq 0$, we have $x(p,\alpha w) = \alpha x(p,w)$. In fact, since $v(\cdot,\cdot)$ is homogeneous of degree one in w, $\nabla_p v(p,\alpha w) = \alpha\nabla_p v(p,w)$ and $\nabla_w v(p,\alpha w) = \nabla_w v(p,w)$. Thus, by Roy's identity, $x(p,\alpha w) = \alpha x(p,w)$.

Now let $x \in \mathbb{R}^L_+$, $x' \in \mathbb{R}^L_+$, $u(x) = u(x')$, and $\alpha \geq 0$. Since $u(\cdot)$ is strictly quasiconcave, by the supporting hyperplane theorem (Theorem M.G.3), there exist $p \in \mathbb{R}^L_{++}$, $p' \in \mathbb{R}^L_{++}$, $w \geq 0$, and $w' \geq 0$ such that $x = x(p,w)$ and $x' = x(p',w')$. Then $u(x) = v(p,w)$ and $u(x') = v(p',w')$. Hence $v(p,w) = v(p',w')$. Thus, by the homogeneity, $v(p,\alpha w) = v(p',\alpha w')$. But as we saw above, $x(p,\alpha w) = \alpha x$ and $x(p',\alpha w') = \alpha x'$. Hence $v(p,\alpha w) = u(\alpha x)$ and $v(p',\alpha w') = u(\alpha x')$. Thus $u(\alpha x) = u(\alpha x')$. Therefore $u(x)$ is homogeneous of degree one.

3.D.6 (a) Define $\tilde{u}(x) = u(x)^{1/(\alpha+\beta+\gamma)} = (x_1 - b_1)^{\alpha'}(x_2 - b_2)^{\beta'}(x_3 - b_3)^{\gamma'}$, with $\alpha' = \alpha/(\alpha + \beta + \gamma)$, $\beta' = \beta/(\alpha + \beta + \gamma)$, $\gamma' = \gamma/(\alpha + \beta + \gamma)$. Then $\alpha' + \beta' + \gamma' = 1$ and $\tilde{u}(\cdot)$ represents the same preferences as $u(\cdot)$, because the function $u \to u^{1/(\alpha+\beta+\gamma)}$ is a monotone transformation. Thus we can assume without loss of generality that $\alpha + \beta + \gamma = 1$.

(b) Use another monotone transformation of the given utility function,

$$\ln u(x) = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3).$$

The first-order condition of the UMP yields the demand function

$$x(p,w) = (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3),$$

where $p\cdot b = p_1b_1 + p_2b_2 + p_3b_3$. Plug this demand function to $u(\cdot)$, then we obtain the indirect utility function

$$v(p,w) = (w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma}.$$

(c) To check the homogeneity of the demand function,

$$x(\lambda p,\lambda w) = (b_1,b_2,b_3) + (\lambda w - \lambda p\cdot b)(\alpha/\lambda p_1,\beta/\lambda p_2,\gamma/\lambda p_3)$$
$$= (b_1,b_2,b_3) + (w - p\cdot b)(\alpha/p_1,\beta/p_2,\gamma/p_3) = x(p,w).$$

To check Walras law,

$$p\cdot x(p,w) = p\cdot b + (w - p\cdot b)(p_1\alpha/p_1 + p_2\beta/p_2 + p_3\gamma/p_3)$$
$$= p\cdot b + (w - p\cdot b)(\alpha + \beta + \gamma) = w.$$

The uniqueness is obvious.

To check the homogeneity of the indirect utility function,

$$v(\lambda p,\lambda w) = (\lambda w - \lambda p\cdot b)(\alpha/\lambda p_1)^{\alpha}(\beta/\lambda p_2)^{\beta}(\gamma/\lambda p_3)^{\gamma}$$
$$= \lambda^{1-(\alpha+\beta+\gamma)}(w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma}$$
$$= (w - p\cdot b)(\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} = v(p,w).$$

To check the monotonicity,

$$\partial v(p,w)/\partial w = (\alpha/p_1)^{\alpha}(\beta/p_2)^{\beta}(\gamma/p_3)^{\gamma} > 0,$$

$$\partial v(p,w)/\partial p_1 = v(p,w)\cdot(-\alpha/p_1) < 0,$$

$$\partial v(p,w)/\partial p_2 = v(p,w)\cdot(-\beta/p_2) < 0,$$

$$\partial v(p,w)/\partial p_3 = v(p,w)\cdot(-\gamma/p_3) < 0.$$

The continuity follows directly from the given functional form. In order to prove the quasiconvexity, it is sufficient to prove that, for any $v \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^3 : v(p,w) \le v\}$ is convex. Consider

$$\ln v(p,w) = \alpha\ln\alpha + \beta\ln\beta + \gamma\ln\gamma + \ln(w - p\cdot b) - \alpha\ln p_1 - \beta\ln p_2 - \gamma\ln p_3.$$

Since the logarithmic function is concave, the set

3.E.4 Suppose first that $\succsim$ is convex and that $x \in h(p,u)$ and $x' \in h(p,u)$. Then $p \cdot x = p \cdot x'$ and $u(x) \geq u$, $u(x') \geq u$. Let $\alpha \in [0,1]$ and define $x'' = \alpha x + (1 - \alpha)x'$. Then $p \cdot x'' = \alpha p \cdot x + (1 - \alpha)p \cdot x' = p \cdot x = p \cdot x'$ and, by the convexity of $\succsim$, $u(x'') \geq u$. Thus $x'' \in h(p,u)$.

Suppose next that $\succsim$ is strictly convex and that $x \in h(p,u)$, $x' \in h(p,u)$, $x \neq x'$, and $u(x) \geq u(x') \geq u$. By the argument above, $x'' = \alpha x + (1 - \alpha)x'$ with $\alpha \in (0,1)$ satisfies $p \cdot x'' = p \cdot x = p \cdot x'$ and, by the strict convexity of $\succsim$, we have $x'' \succ x'$. Since $\succsim$ is continuous, $\beta x'' \succ x'$ for any $\beta \in (0,1)$ close enough to 1. But this implies that $p \cdot (\beta x'') < p \cdot x$ and $u(\beta x'') > u(x') \geq u$, which contradicts the fact that x is a solution of the EMP. Hence $h(p,u)$ must be a singleton.

3.E.9 First, we shall prove that Proposition 3.D.3 implies Proposition 3.E.2 via (3.E.1). Let $p \gg 0$, $p' \gg 0$, $u \in \mathbb{R}$, $u' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) Homogeneity of degree one in p: Let $\alpha > 0$. Define $w = e(p,u)$, then $u = v(p,w)$ by the second relation of (3.E.1). Hence

$$e(\alpha p,u) = e(\alpha p,v(p,w)) = e(\alpha p,v(\alpha p,\alpha w)) = \alpha w = \alpha e(p,u),$$

where the second equality follows from the homogeneity of $v(\cdot,\cdot)$ and the third from the first relation of (3.E.1).

(ii) Monotonicity: Let $u' > u$. Define $w = e(p,u)$ and $w' = e(p,u')$, then $u = v(p,w)$ and $u' = v(p,w')$. By the monotonicity of $v(\cdot,\cdot)$ in w, we must have $w' > w$, that is, $e(p',u) > e(p,u)$.

Next let $p' \geq p$. Define $w = e(p,u)$ and $w' = e(p',u)$, then, by the second relation of (3.E.1), $u = v(p,w) = v(p',w')$. By the monotonicity of $v(\cdot,\cdot)$, we must have $w' \geq w$, that is, $e(p',u) \geq e(p,u)$.

(iii) Concavity: Let $\alpha \in [0,1]$. Define $w = e(p,u)$ and $w' = e(p',u)$, then $u = v(p,w) = v(p',w)$. Define $p'' = \alpha p + (1 - \alpha)p''$ and $w'' = \alpha w + (1 - \alpha)w'$. Then, by the quasiconvexity of $v(\cdot,\cdot)$, $v(p'',w'') \leq u$. Hence, by the monotonicity of $v(\cdot,\cdot)$ in w and the second relation of (3.E.1), $w'' \leq e(p'',u)$. that is,

$$e(\alpha p + (1 - \alpha)p',\ u) \geq \alpha e(p,u) + (1 - \alpha)e(p',u).$$